International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 9, September 2015, PP 35-48
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Some New Fixed Point and Common Fixed Point Theorems in Dislocated Metric Spaces and Dislocated Quasi -Metric Spaces

Manoj Ughade[1], R. D. Daheriya[2]

[1]Department of Mathematics, Sarvepalli Radhakrishnan University, Bhopal, Madhya Pradesh, India

[2]Department of Mathematics, Government J.H. Post Graduate College, Betul, Madhya Pradesh, India

Abstract: *Our goal in this paper is to study of fixed point theorems in dislocated quasi-metric spaces and dislocated metric spaces which we can apply in a variety of different situations. This article can be considered as a continuation of the remarkable works of Hitzler et al [9], Zeyada et al. [6] and Geraghty [17]. In this article, we review briefly some generalizations of metric space with examples and we describe some properties, introduce new definitions and present some lemmas and propositions related to dislocated metric spaces and dislocated quasi-metric spaces. We also present some fixed point and common fixed point theorems for self-mappings in a complete dislocated metric spaces and quasi-metric spaces under various contractive conditions and present some examples to illustrate the effectiveness of our results.*

Keywords and Phrases: *Fixed point, Common fixed point, dislocated metric spaces, dislocated quasi-metric spaces, contraction.*

MATHEMATICS SUBJECT CLASSIFICATION (2010): 47H10, 54H25;

1. Introduction and Preliminares

Fixed point theory is one of the most dynamic research subjects in nonlinear sciences. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. One of the simplest and most useful results in fixed point theory is the Banach fixed point theorem: Let (X, d)be a complete metric space and T be self mapping of X satisfying

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

where $\lambda \in [0,1)$, then T has a unique fixed point. A mapping satisfying the condition (1.1) is called contraction mapping. As well as, there are a lot of extensions of this famous fixed point theorem in metric space which are obtained generalizing contractive condition, there are a lot of generalizations of it in different space which has metric type structure. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique. This elementary technique leads to increasing of the possibility of solving various problems in different research fields. This celebrated result has been generalized in many abstract spaces for distinct operators.

The concept of dislocated metric space was introduced by P. Hitzler [3] in which the self distance of points need not to be zero necessarily. They also generalized famous Banach's contraction principle in dislocated metric space. Dislocated metric space play a vital rule in topology, logical programming and electronic engineering. F. M. Zeyada et al.[6] develops the notation of complete dislocated quasi metric spaces and generalized the result of Hitzler [3] in dislocated quasi metric space. In [2] C.T. Aage and J. N. Salunke proved dislocated quasi-metric version of Kannan mapping theorem. Amini-Haradi [7] re-introduced the dislocated space under the name of a metric-like space and proved some fixed theorems in this space. Very recently, Bennani et al. [8] established two new common fixed point theorems for four self maps on dislocated metric spaces, which improved the results of Panthi and Jha [10] without any continuity requirement. After F. M. Zeyada et al.[6] many papers have been published containing fixed point results in dislocated quasi metric spaces (see [1, 2, 4, 5, 11,12,13,14, 29, 30, 31]).

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}_0^+ = [0, \infty)$. Now, we will discuss difference between metric space and generalizations of metric space (quasi-metric, pseudo-metric, dislocated metric, dislocated quasi-metric, partial metric, ultra metric).

The following definition will be needed in the sequel.

Definition 1.1 Consider a non-empty set X, whose elements will be refered to as points. A distance function on a set X is a function $d: X \times X \to \mathbb{R}_0^+$ which assigns to each pair of points x and y in X a real number $d(x, y)$.We need the following conditions:

(1) Non-negativity or Separation Axiom: $\forall\ x, y \in X,\ d(x, y) \geq 0$;

(2) Identity of Indiscernible or Faithful : $\forall\ x, y \in X,\ d(x, y) = 0 \Leftrightarrow x = y$;

(3) Small self-distance: $\forall\ x \in X, d(x, x) = 0$; (But possibly $d(x, y) = 0$ for some distinct values $x \neq y$).

(4) Indistancy Implies Equality: $\forall\ x, y \in X,\ d(x, y) = 0 = d(y, x) \Rightarrow x = y$;

(5) Equality : $\forall\ x, y \in X,\ x = y \Leftrightarrow d(x, x) = d(x, y) = d(x, y)$;

(6) Small self-distances: $\forall\ x, y \in X,\ d(x, x) \leq d(x, y)$;

(7) Symmetry: $\forall\ x, y \in X,\ d(x, y) = d(y, x)$;

(8) The Triangle Inequality: $\forall\ x, y, z \in X,\ d(x, y) \leq d(x, z) + d(z, y)$;

(9) The Triangle Inequality: $\forall\ x, y, z \in X,\ d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$;

(10) Strong Triangle Inequality: $\forall\ x, y, z \in X,\ d(x, y) \leq \max\{d(x, z), d(z, y)\}$;

If d satisfies conditions (1), (2), (7) and (8), then it is called a metric on X. If it satisfies conditions (1), (2), and (8), it is called a quasi-metric on X. If a metric d satisfies the strong triangle inequality (10), then it is called an ultra metric. If it satisfies conditions (1), (5), (6) and (9), it is called a partial metric. If it satisfies conditions (1), (4), (7) and (8), it is called a dislocated metric (or simply d-metric) on X and the pair (X, d) is called a dislocated metric space. Moreover "$d(x, y) = 0 \Rightarrow x = y$" when d is a d-metric. However "$x = y$" does not necessarily imply "$d(x, y) = 0$" when d is a d-metric. If d is a d-metric instead of a metric, it is possible that $d(x, x) \neq 0$.As such these implications hold well in a d-metric space as well when "$x \neq y$" is replaced by $d(x, y) \neq 0$. If it satisfies conditions (1), (4), and (8), it is called a dislocated quasi-metric (or simply d_q-metric).

Clearly every metric space is a dislocated metric space but the converse is not necessarily true as clear form the following example.

Example 1.2 Let $X = [0,1]$ define the distance function $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = \max\{x, y\}$. Clearly X is dislocated metric space but not a metric space.

Also every metric space is dislocated quasi metric space but the converse is not true and every dislocated metric space is dislocated quasi metric space but the converse is not true as clear from the following example.

Example 1.3 Let $X = [0, 1]$, we define the function $d : X \times X \to \mathbb{R}^+$ as $d(x, y) = |x - y| + |x|$ for all $x, y \in X$. Clearly X is dq-metric space but not a metric space nor dislocated metric space.

In our main work we will use the following definitions which can be found in [3, 6].

Definition 1.4 A sequence (x_n) in d_q-metric space (X, d) with respect to d_q is said to be d_q - converge to $x \in X$ provided that

$$d_q - \lim_{n\to\infty} d(x_n, x) = d_q - \lim_{n\to\infty} d(x, x_n) = 0 .$$

In this case x is called the d_q −limit of (x_n) and we write $d_q - \lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.5 A sequence (x_n) in d −metric space (X, d) with respect to d is said to be d –converge to $x \in X$ provided that

$$d - \lim_{n\to\infty} d(x_n, x) = d - \lim_{n\to\infty} d(x, x_n) = 0 .$$

In this case x is called the d −limit of (x_n) and we write $d - \lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.6 We call a sequence (x_n) in d_q-metric space (d-metric space) (X, d) is a d_q-Cauchy (d -Cauchy) sequence provided that for all $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon, \forall\, m, n \geq n_0$.

Definition 1.7 A d_q-metric space (d-metric space) (X, d) is called d_q-complete (d-complete) if every d_q-Cauchy (d-Cauchy) sequence in X converges with respect to d in X.

We discuss a few more examples of quasi-metric spaces, d-metrics and d_q-metrics.

Example 1.8 Let X be the set of positive integers and define the non-negative real valued function $d: X \times X \to \mathbb{R}_0^+$ by

$$d(x,y) = \begin{cases} \frac{1}{y}, & \text{if } x < y \\ 0, & \text{if } x = y \\ 1, & \text{if } x > y \end{cases}$$

Note that d satisfies all the axioms of quasi-metric and therefore the pair (X, d) is a quasi-metric space but it is not a metric space because d is non-symmetric, i.e. $d(x, y) \neq d(y, x)$.

Example 1.9 Let $X = \mathbb{R}$. Define $d: X \times X \to \mathbb{R}_0^+$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ |y| & \text{otherwise,} \end{cases}$$

Then one can easily see that d is a quasi-metric and the pair (X, d) is a quasi-metric space.

Example 1.10 Let $X = [0,1]$. Define $d: X \times X \to \mathbb{R}_0^+$ by

1. $d(x,y) = \begin{cases} y - x, & \text{if } x \leq y, \\ 2(x - y), & \text{otherwise,} \end{cases}$
2. $d(x,y) = \begin{cases} x - y, & \text{if } x \geq y, \\ \frac{1}{2}(y - x), & \text{otherwise,} \end{cases}$

Then here also d is a quasi-metric and the pair (X, d) is a quasi-metric space.

Example 1.11 Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ by $d(x, y) = |x| + y^2$, then d is d_q- metric on $\mathbb{R}$ which is not a d - metric and the pair $(\mathbb{R}, d)$ is a d_q- metric space.

Example 1.12 Let $X = \mathbb{R}_0^+$ and define a distance function $d: X \times X \to \mathbb{R}_0^+$ such that $d(x, y) = \max\{x, y\}$. Then d is a d-metric on X and the pair (X, d) is a d-metric space.

Example 1.13 Let $X = \left\{\frac{1}{2^n} \mid n \in \mathbb{N} \cup \{0\}\right\}$. Define $d: X \times X \to R_0^+$ by setting $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then d is a d-metric on X and the pair (X, d) is a d-metric space.

Example 1.14 Let $X = \mathbb{R}_0^+$. Then $d: X \times X \to \mathbb{R}_0^+$ define by $d(x, y) = \frac{1}{2}|x - y| + \frac{1}{2}(x + y)$. for all $x, y \in R_0^+$ is a d-metric on R_0^+ and the pair (X, d) is a d-metric space.

Example 1.15 Let I be the set of all closed interval on $\mathbb{R}$. Then $d: I \times I \to \mathbb{R}_0^+$ defined by $d([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ for all $[a, b], [c, d] \in I$, is a d-metric on X and (I, d) is a d-metric space.

Example 1.16 Let $X = \{0,1,2\}$ and a mapping $d: X \times X \to \mathbb{R}_0^+$ be defined as

$$d(0,0) = d(1,1) = d(2,2) = 0,$$

$$d(0,1) = d(1,2) = d(0,2) = 1,$$

$$d(1,0) = d(2,1) = d(2,0) = 2.$$

Note that d satisfies all the axioms of d_q-metric and the pair (X, d) is a d_q-metric space but it is not a d-metric space because d is non-symmetric i.e. $d(x, y) \neq d(y, x)$.

Example 1.17 Let $X = \mathbb{R}_0^+$ and define the function $d: X \times X \to \mathbb{R}_0^+$ by $d(x, y) = |x - y| + |x|$ for all $x, y \in X$ evidently d is d_q-metric but not a metric nor d-metric.

Example 1.18 Let $X = \{0,1\}$ and a mapping $d: X \times X \to \mathbb{R}_0^+$ be defined as $d(1,1) = d(2,2) = 0, d(1,2) = d(2,1) = 1$. We observe that d satisfies all the axioms of d-metric and the pair (X, d) is a d-metric space. Also (X, d) is a d_q-metric space.

Definition 1.19 Let S and T be two self-mappings on a nonempty set X; then

1) any point $x \in X$ is said to be fixed point of T if $Tx = x$;
2) any point $x \in X$ is called coincidence point of S and T if $Sx = Tx$ and one calls $u = Sx = Tx$ a point of coincidence of S and T;
3) a point $x \in X$ is called common fixed point of S and T if $Sx = Tx = x$.

Lemma 1.20 (see [6]) Let (X, d) be a dq-metric space. If $f : X \to X$ is a contraction function, then $f^n(x_0)$ is a Cauchy sequence for each $x_0 \in X$.

Lemma 1.21 (see [6]) Every subsequence of d_q-convergent sequence to a point x_0 is d_q- convergent to x_0.

Theorem 1.22 (see [6]) Let (X, d) be a d_q-metric space and let $T: X \to X$ be continuous contraction mapping. Then T has a unique fixed point.

Theorem 1.23 (see [19]) Let (X, d) be a complete d-metric space and let $T: X \to X$ be a contraction. Then T has a unique fixed point.

2. MAIN RESULT

In this section, first we explore the properties of dislocated metric space and dislocated quasi-metric space.

Proposition 2.1 Every converging sequence in a d-metric space is a Cauchy sequence.

Proof Let (x_n) be a sequence which converges to some x, and let $\epsilon > 0$ be arbitrarily chosen. Then there exists $n_0 \in \mathbb{N}$ with $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq n_0$. For $m, n \geq n_0$, we then obtain $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence (x_n) is a Cauchy sequence.

Proposition 2.2 Limits in d-metric spaces are unique.

Proof Let x^* and y^* be limits of the sequence $\{x_n\}$ .Then $d(x_n, x^*) \to 0$ and $d(x_n, y^*) \to 0$ as $n \to \infty$. By the triangle inequality, it follows that $d(x^*, y^*) \leq d(x^*, x_n) + d(x_n, y^*) = d(x_n, x^*) + d(x_n, y^*) \to 0$ as $n \to \infty$. Hence $d(x^*, y^*) = 0$ and so $x^* = y^*$.

Proposition 2.3 Limits in d_q-metric spaces are unique.

Proof Let x^* and y^* be limits of the sequence (x_n) .Then $x_n \to x^*$ and $x_n \to y^*$ as $n \to \infty$. By the triangle inequality, it follows that $d(x^*, y^*) \leq d(x^*, x_n) + d(x_n, y^*)$. Letting $n \to \infty$, we obtain $d(x^*, y^*) \leq d(x^*, x^*) + d(y^*, y^*)$. Similarly $d(y^*, x^*) \leq d(x^*, x^*) + d(y^*, y^*)$. Hence $|d(x^*, y^*) - d(y^*, x^*)| \leq 0$ and so $d(x^*, y^*) = d(y^*, x^*)$. Also $d(x^*, y^*) \leq d(x^*, x_n) + d(x_n, y^*) \to 0$ as $n \to \infty$ and therefore $d(x^*, y^*) = d(y^*, x^*) = 0$. It follows that $x^* = y^*$.

Now, we introduce the following.

Definition 2.4 Let (X, d) is a d-metric space. Given a point $x_0 \in X$ and a real number $\epsilon > 0$. We define three types of sets.

Open Ball: $\mathcal{B}_\epsilon^d(x_0) = \{ x \in X \mid d(x_0, x) < \epsilon \}$

Closed Ball: $\widetilde{\mathcal{B}}_\epsilon^d(x_0) = \{ x \in X \mid d(x_0, x) \leq \epsilon \}$

Sphere: $\mathcal{S}_\epsilon^d(x_0) = \{ x \in X \mid d(x_0, x) = \epsilon \}$

In all the above three cases x_0 is called the centre and ϵ the radius. We observe that

$$\mathcal{S}_\epsilon^d(x_0) = \widetilde{\mathcal{B}}_\epsilon^d(x_0) - \mathcal{B}_\epsilon^d(x_0).$$

If (X, d) is a d_q-metric space. Given a point $x_0 \in X$ and a real number $\epsilon > 0$. Then

Open Ball: $\mathcal{B}_\epsilon^{d_q}(x_0) = \{ x \in X \mid \min\{d(x_0, x), d(x, x_0)\} < \epsilon \}$

Closed Ball: $\widetilde{\mathcal{B}}_\epsilon^{d_q}(x_0) = \{ x \in X \mid \min\{d(x_0, x), d(x, x_0)\} \leq \epsilon \}$

Sphere: $\mathcal{S}_\epsilon^{d_q}(x_0) = \{ x \in X \mid \min\{d(x_0, x), d(x, x_0)\} = \epsilon \}$

In all the above three cases x_0 is called the centre and ϵ the radius. Note that

$$\mathcal{S}_\epsilon^{d_q}(x_0) = \widetilde{\mathcal{B}}_\epsilon^{d_q}(x_0) - \mathcal{B}_\epsilon^{d_q}(x_0).$$

There is no guarantee that $x \in \mathcal{B}_\epsilon^{d_q}(x)$, for example, in example 1.14, if $x > 0$ and $y > 0$, then $1 \notin \mathcal{B}_{\frac{1}{4}}^{d_q}$.

Definition 2.5 Neighbourhood is an (open $\epsilon -$) ball in a d-metric space (X, d) with center $x \in X$ is a set $\mathcal{N}_\epsilon^d(x) = \{y \in X \mid d(x, y) < \epsilon\}$ where $\epsilon > 0$.

Note that an (open $\epsilon -$) ball may be empty in d-metric space. In fact, the centre of an (open $\epsilon -$) ball is contained in the (open $\epsilon -$) ball itself; the point may be dislocated from the ball.

In a d_q-metric space (X, d), a neighbourhood is an (open $\epsilon -$) ball with center $x \in X$ is a set $\mathcal{N}_\epsilon^{d_q}(x) = \{y \in X \mid \min\{d(x, y), d(y, x)\} < \epsilon\}$ where $\epsilon > 0$.

Now, we give the following two Propositions.

Proposition 2.6 Let (X, d) be a dislocated metric space. Then for all $x \in X$. $d(x, x) = 0$ iff $\mathcal{N}_\epsilon^d(x) \neq \emptyset$. for all $\epsilon > 0$ and $x \in X$.

Proof Since $\mathcal{N}_\epsilon^d(x) = \{y \in X \mid d(x, y) < \epsilon\}$, where $\epsilon > 0$, is an open $\epsilon -$ball in a d-metric space (X, d) with center $x \in X$. Suppose small self distance: $d(x, x) = 0$ for all $x \in X$ and $\epsilon > 0$ be given, then $d(x, x) < \epsilon \Rightarrow x \in \mathcal{N}_\epsilon^d(x)$. Hence $\mathcal{N}_\epsilon^d(x) \neq \emptyset$ for all and $x \in X$. Conversely, suppose that $\mathcal{N}_\epsilon^d(x) \neq \emptyset$ for all $\epsilon > 0$. We know that $\mathcal{N}_\epsilon^d(x) = \{y \in X \mid d(x, y) < \epsilon\}$. Then $\exists\ y \in X$ such that $d(x, y) = r < \epsilon$. By using triangle inequality and symmetry property, it follows that $d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y) = 2r$ and hence for all $\epsilon > 0$, $d(x, x) < \epsilon$. Therefore $d(x, x) = 0$.

Proposition 2.7 Let (X, d) be a d-metric space. Then d is a metric if and only if for all $x \in X$ and all $\varepsilon > 0$, $\mathcal{N}_\epsilon^d(x) \neq \emptyset$.

Proof Let d is a metric. We then have $d(x, x) = 0\ \forall\, x \in X$. By Proposition (2.6), it follows that $\mathcal{N}_\epsilon^d(x) \neq \emptyset$. Conversely, let $\mathcal{N}_\epsilon^d(x) \neq \emptyset$ for all $x \in X$ and all $\varepsilon > 0$. Since $\mathcal{N}_\epsilon^d(x) = \{y \in X | d(x, y) < \epsilon\}$. Then $d(x, x) < \epsilon$. Hence for all $x \in X$, $d(x, x) = 0$, and then $d(x, y) = 0$ if and only if $x = y$. Also symmetric and triangular properties hold. Hence d is a metric.

Definition 2.8 Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be two metric spaces. A mapping $f: X \to Y$ is said to be continuous at a point $x_0 \in X$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that $\tilde{d}(fx, fx_0) < \epsilon$ for all x satisfying $d(x, x_0) < \delta$. Mapping f is said to be continuous, if it is continuous at every point of X.

Theorem 2.9 A mapping $f: X \to Y$ of a d-metric space (X, d) into a d-metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0$ implies that $fx_n \to fx_0$.

Proof Assume f to be continuous at x_0. Then for given $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $\tilde{d}(fx, fx_0) < \epsilon$. Let (x_n) be a sequence in X such that $x_n \to x_0$. Then there is a n_0 such that for all $n > n_0$, we have $d(x_n, x_0) < \delta$. Hence for all $n > n_0$, $\tilde{d}(fx_n, fx_0) < \epsilon$. By definition this means that $fx_n \to fx_0$. Conversely we assume that T is continuous at x_0. suppose this is false. Then there is an $\epsilon > 0$ such that for every $\delta > 0$, there is an $x \neq x_0$ satisfying $d(x, x_0) < \delta$ but $\tilde{d}(fx, fx_0) \geq \epsilon$.

In particular for $\delta = \frac{1}{n}$, there is an x_n satisfying $d(x_n, x_0) < \frac{1}{n}$, but $\tilde{d}(fx_n, fx_0) \geq \epsilon$. Clearly $x_n \to x_0$ but (fx_n) does not converge to fx_0. This contradicts $fx_n \to fx_0$ and proves the theorem.

Now, we are ready to state and prove our first main results.

Theorem 2.10 Let $\tau: \mathbb{R}_0^+ \to [0, 2^{-1})$ be decreasing function. Let (X, d) be a complete d-metric space and let $T: X \to X$ be a self-mapping such that

$$d(Tx, Ty) \leq \tau\big(d(x, y)\big)\big(d(x, Tx) + d(y, Ty)\big) \tag{2.1}$$

for all $x, y \in X$. If moreover, T is continuous or τ is a constant function then T has a unique fixed point $x^* \in X$.

Proof Let x_0 be an arbitrary point in X and we set $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$. We may assume that $d(x_n, x_{n+1}) \neq 0$ for each $n \in \mathbb{N}_0$. Then by (2.1), we get

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \tau\big(d(x_n, x_{n+1})\big)\big(d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\big) \\ &= \tau\big(d(x_n, x_{n+1})\big)\big(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big) \\ &\leq 2^{-1}\big(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big) \end{aligned} \tag{2.2}$$

Hence, $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ for each $n \in \mathbb{N}_0$. So $\{d(x_n, x_{n+1})\}$ is a nonnegative decreasing sequence. Hence, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} d(x_n\, x_{n+1}) = \gamma$. We claim that $\gamma = 0$. Suppose, on the contrary, that $\gamma > 0$. Then due to (2.2), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \tau(\gamma)\big(d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\big) \\ &= \tau(\gamma)\big(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big) \end{aligned} \tag{2.3}$$

and consequently $\gamma \leq 2\tau(\gamma)\gamma$, which is impossible since $\tau(\gamma) < 2^{-1}$. This proves that $\gamma = 0$. Now we will show that (x_n) is a Cauchy sequence. From (2.1), we get

$$\begin{aligned} d(x_n, x_m) &= d(fx_n, fx_m) \\ &\leq \tfrac{1}{2}[d(x_n, fx_n) + d(x_m, fx_m)] \\ &= \tfrac{1}{2}[d(x_n, x_{n+1}) + d(x_m, x_{m+1})] \end{aligned} \tag{2.4}$$

There exist an n_0 such that for $m, n \geq n_0$, $d(x_n, x_{n+1}) < \epsilon$ and $d(x_m, x_{m+1}) < \epsilon$. Hence

$$d(x_n, x_m) \leq \tfrac{1}{2}(\epsilon + \epsilon) = \epsilon \tag{2.5}$$

for all $m, n \geq n_0$. This forces that (x_n) is Cauchy sequence and view of completeness of X, there exists an $x^* \in X$ such that $x_n \to x^*$. We can check that $Tx^* = x^*$. If T is continuous, then

$$\begin{aligned} Tx^* &= T(lim_{n\to\infty}\, x_n) = lim_{n\to\infty}\, Tx_n \\ &= lim_{n\to\infty}\, x_{n+1} = x^* \end{aligned} \tag{2.6}$$

and x^* is a fixed point of T. On the other hand, if τ is a constant. Then, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \tau\,\big(d(x_n, x_{n+1}) + d(x^*, Tx^*)\big) \end{aligned} \tag{2.7}$$

Letting $n \to \infty$, we get

$$d(x^*, Tx^*) \leq \tau\, d(x^*, Tx^*). \tag{2.8}$$

Similarly,

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, x_{n+1}) + d(x_{n+1}, x^*) \\ &= d(Tx^*, Tx_n) + d(x_{n+1}, x^*) \end{aligned} \tag{2.9}$$

$$\leq \tau\left(d(x^*, Tx^*) + d(x_n, Tx_n)\right) + d(x_{n+1}, x^*)$$
$$= \tau\left(d(x^*, Tx^*) + d(x_n, x_{n+1})\right) + d(x_{n+1}, x^*)$$

Letting $n \to \infty$, we get

$$d(Tx^*, x^*) \leq \tau\left(d(x^*, Tx^*)\right) \tag{2.10}$$

Hence $|d(x^*, Tx^*) - d(Tx^*, x^*)| \leq 0$. Thus, $d(x^*, Tx^*) = 0 = d(Tx^*, x^*)$ and so $Tx^* = x^*$. Finally, to prove the last part of the theorem. Let us assume that $Tx^* = x^*,\ Ty^* = y^*,\ y^* \neq x^*,\ x^*, y^* \in X$. First we show that $d(x^*, x^*) = d(y^*, y^*) = 0$. From condition (2.1), we have

$$d(x^*, x^*) \leq \tau\big(d(x^*, x^*)\big)\big(d(x^*, x^*) + d(x^*, x^*)\big) \tag{2.11}$$
$$= 2\tau\big(d(x^*, x^*)\big)d(x^*, x^*)$$

Since $\tau: \mathbb{R}_0^+ \to [0, 2^{-1})$ be decreasing function, so the above inequality is possible if $d(x^*, x^*) = 0$. Similarly we can show that $d(y^*, y^*) = 0$. Now we consider

$$d(x^*, y^*) = d(\mathrm{T}x^*, Ty^*) \tag{2.12}$$
$$\leq \tau\big(d(x^*, y^*)\big)\big(d(x^*, T\,x^*) + d(y^*, Ty^*)\big)$$
$$\leq \tau\big(d(x^*, y^*)\big)\big(d(x^*, x^*) + d(y^*, y^*)\big)$$

This shows that $d(x^*, y^*) = 0$ and $x^* = y^*$, which proves the uniqueness of fixed point of T.

Now, we furnished an example to support our main result (Theorem 2.10).

Example 2.11 Let $X = [0,1]$. Define $d: X \times X \to \mathbb{R}_0^+$ by$d(x, y) = x + y$. Then d is dislocated metric on X and the pair (X, d) is a complete dislocated metric space. Also, we define $\tau: [0, \infty) \to [0, 2^{-1})$ by $\tau(t) = (t + 4)^{-1}$ and $T: X \to X$ by $T(x) = \frac{x}{7}$. Obviously, τ is a nonnegative decreasing function. Also the map T is continuous in X. For all $x, y \in X$, we obtain

$$d\big(f(x), f(y)\big) \leq \tau\big(d(x, y)\big)\big(d(x, fx) + d(y, fy)\big)$$

$$\Rightarrow \qquad d\left(\frac{x}{7}, \frac{y}{7}\right) \leq \tau(x + y).\left(d\left(x, \frac{x}{7}\right) + d\left(y, \frac{7}{7}\right)\right)$$

$$\Rightarrow \qquad \frac{x}{7} + \frac{y}{7} \leq \frac{1}{x+y+4} \cdot \left(x + \frac{x}{7} + y + \frac{y}{7}\right)$$

$$\Rightarrow \qquad \frac{x+y}{7} \leq \frac{8}{7}\frac{(x+y)}{x+y+4} \quad \Rightarrow \quad 1 \leq 8\left(\frac{1}{x+y+4}\right)$$

$$\Rightarrow \quad x + y + 4 \leq 8 \qquad \Rightarrow \qquad x + y \leq 4\,, \quad \forall\, x, y \in X.$$

Clearly T satisfies the condition (2.1). Thus T satisfies all the hypotheses of Theorem 2.10 and $x^* = 0$ is the unique fixed point of T.

In [25] Jungck introduced the concept of commuting maps. In [26] Jungck introduced the concept of compatible mappings which generalize the concept of commuting maps. Jungck in [27] further generalized the concept of compatible maps as follows.

Definition 2.12 Let S and T be two mappings from a metric space (X, d) into itself. Then, S and T are said to be weakly compatible if they commute at their coincidence points; that is, $Sx = Tx$ for some $x \in X$ implies $STx = TSx$.

Now, we have the following key lemma.

Lemma 2.13 Let X be a non-empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and $(T, f\,)$ are weakly compatible, then S, T and f have a unique common fixed point.

Proof Since v is point of coincidence S, T and f. Therefore, $v = fu = Su = Tu$ for some $u \in X$. By weakly compatibility of $(S, f\,)$ and $(T, f\,)$ we have $Sv = Sfu = fSu = fv$ and$Tv = Tfu =$

$fTu = fv$. It implies that $Sv = Tv = fv = w$ (say). Then w is a point of coincidence of S, T and f. Therefore, $v = w$ by uniqueness. Thus v is a unique common fixed point of S, T and f .

Our next theorem is about a common fixed point for three self-mapping satisfying contraction type condition in the context of d-metric space.

Theorem 2.14 Let (X, d) be a dislocated metric space and the mappings $S, T, f : X \to X$ satisfy:

$$d(Sx, Ty) \leq \lambda d(fx, fy) \tag{2.13}$$

for all $x, y \in X$ where $0 \leq \lambda < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof Let $x_0 \in X$ be arbitrary. Choose a point x_1 in X such that $fx_1 = Sx_0$. This can be done since $S(X) \subseteq f(X)$. Similarly, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_{2k+1} = Sx_{2k}, \ fx_{2k+2} = Tx_{2k+1}, \ k = 0, 1, 2, \dots. \tag{2.14}$$

From (2.13), we get

$$d(fx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \leq \lambda d(fx_{2k}, fx_{2k+1}) \tag{2.15}$$

Similarly,

$$d(fx_{2k+2}, fx_{2k+3}) = d(Sx_{2k+1}, Tx_{2k+2}) \leq \lambda d(fx_{2k+1}, \mathrm{f}x_{2k+2}) \tag{2.16}$$

Now by induction, we obtain for each $k = 0, 1, 2, \dots.$

$$d(fx_{2k+2}, fx_{2k+3}) \leq \lambda^{2k+2} d(fx_0, fx_1) \tag{2.17}$$

Let

$$y_n = fx_n, \ n = 0, 1, 2, \dots \tag{2.18}$$

Now for all n, we have

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \leq \cdots \leq \lambda^{n+1} d(y_0, y_1) \tag{2.19}$$

Now for any m > n,

$$\begin{aligned} d(y_m, y_n) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(y_0, y_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(y_0, y_1) \end{aligned} \tag{2.20}$$

Assume that $d(y_0, y_1) > 0$. Letting $n \to +\infty$, (y_n) is a Cauchy sequence. Also, if $d(y_0, y_1) = 0$, then $d(y_m, y_n) = 0$ for all $m > n$ and hence (y_n) is a Cauchy sequence in X. Since $f(X)$ is complete, there exists $u, v \in X$ such that $y_n \to v = fu$. Now, we show that v is a common point of coincidence of S, T and f that is $v = fu = Su = Tu$.

$$\begin{aligned} d(fu, Su) &\leq d(fu, y_{2n+2}) + d(y_{2n+2}, Su) \\ &\leq d(v, y_{2n+2}) + d(Tx_{2n+1}, Su) \\ &\leq d(v, y_{2n+2}) + \lambda \, d(fx_{2n+1}, fu) \\ &\leq d(v, y_{2n+2}) + \lambda \, d(y_{2n+1}, v) \ \to 0 \text{ as } n \to \infty \end{aligned} \tag{2.21}$$

Therefore, $d(fu, Su) = 0$. Similarly,

$$\begin{aligned} d(Su, fu) &\leq d(Su, y_{2n+2}) + d(y_{2n+2}, fu) \\ &\leq d(Su, Tx_{2n+1}) + d(y_{2n+2}, v) \\ &\leq \lambda d(fu, fx_{2n+1}) + d(y_{2n+2}, v) \\ &\leq \lambda d(v, y_{2n+1}) + d(y_{2n+2}, v) \ \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.22}$$

Hence $fu = Su$. Similarly, by using

$$d(fu,Tu) \leq d(fu, y_{2n+1}) + d(y_{2n+1},Tu), \tag{2.23}$$

we can show that $fu = Tu$, it implies that v is a common point of coincidence of S, T and f. Now we show that f, S and T have unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$ for some u^* in X. Now,

$$d(v,v^*) = d(Su,Tu^*) \leq \lambda d(fu,fu^*) \leq \lambda d(v,v^*) \tag{2.24}$$

and

$$d(v^*,v) = d(Su^*,Tu) \leq \lambda d(fu^*,fu) \leq \lambda d(v^*,v). \tag{2.25}$$

This implies that $|d(v,v^*) - d(v^*,v)| \leq \lambda|d(v,v^*) - d(v^*,v)|$ that is, $(1-\lambda)\,|d(v,v^*) - d(v^*,v)| \leq 0$. Thus $d(v,v^*) = d(v^*,v) = 0$ and so $v^* = v$. This implies that v^*= v. If (S,f) and (T,f) are weakly compatible, by Lemma 2.13, S, T and f have a unique common fixed point.

Put $S = T$ and $f = I$ an identity mapping in above theorem 2.14 yields Theorem 1.22. Taking $S = T$ in theorem 2.14 yields Corollary 2.15

Corollary 2.15 Let (X, d) be a complete dislocated metric space and the mapping $T : X \to X$ satisfy:

$$d(Tx,Ty) \leq \lambda d(fx,fy) \tag{2.26}$$

for all $x, y \in X$ where $0 \leq \lambda < 1$. If $T(X) \subseteq f(X)$ and f(X) is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if (T, f) is weakly compatible, then T and f have a unique common fixed point.

Theorem 2.16 Let (X, d) be a dislocated metric space and the mappings $S, T, f : X \to X$ satisfy:

$$d(Sx,Ty) \leq \lambda(d(fx,Sx) + d(fy,Ty)) \tag{2.27}$$

for all $x, y \in X$ where $0 \leq \lambda < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and f(X) is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof Let $x_0 \in X$ be arbitrary. Choose a point x_1 in X such that $fx_1 = Sx_0$. This can be done since $S(X) \subseteq f(X)$. Similarly, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process and having chosen x_n in X. We obtain x_{n+1} in X such that

$$fx_{2k+1} = Sx_{2k}, \;\; fx_{2k+2} = Tx_{2k+1}, \; k = 0, 1, 2, \dots. \tag{2.28}$$

From (2.27) we get

$$\begin{aligned} d(fx_{2k+1},fx_{2k+2}) &= d(Sx_{2k},Tx_{2k+1}) \\ &\leq \lambda(d(fx_{2k},Sx_{2k}) + d(fx_{2k+1},Tx_{2k+1})) \\ &= \lambda(d(fx_{2k},fx_{2k+1}) + d(fx_{2k+1},fx_{2k+2})) \end{aligned} \tag{2.29}$$

That is, $\quad d(fx_{2k+1},fx_{2k+2}) \leq \frac{\lambda}{1-\lambda} d(fx_{2k},fx_{2k+1})$

Similarly,

$$\begin{aligned} d(fx_{2k+2},fx_{2k+3}) &= d(Sx_{2k+1},Tx_{2k+2}) \\ &\leq \lambda(d(fx_{2k+1},Sx_{2k+1}) + d(fx_{2k+2},Tx_{2k+2})) \\ &= \lambda(d(fx_{2k+1},fx_{2k+2}) + d(fx_{2k+2},fx_{2k+3})) \end{aligned} \tag{2.30}$$

That is, $\quad d(fx_{2k+2},fx_{2k+3}) \leq \frac{\lambda}{1-\lambda} d(fx_{2k+1},fx_{2k+2})$.

Now by induction, we obtain for each $k = 0, 1, 2, \dots.$

$$\begin{aligned} d(fx_{2k+1},fx_{2k+2}) &\leq \frac{\lambda}{1-\lambda} d(fx_{2k},fx_{2k+1}) \\ &\leq \left(\frac{\lambda}{1-\lambda}\right)^2 d(fx_{2k-1},fx_{2k}) \\ &\vdots \end{aligned} \tag{2.31}$$

$$\leq \left(\frac{\lambda}{1-\lambda}\right)^{2k+1} d(fx_0, fx_1)$$

and

$$d(fx_{2k+1}, fx_{2k+2}) \leq \left(\frac{\lambda}{1-\lambda}\right)^{2k+2} d(fx_0, fx_1) \tag{2.32}$$

Let

$$y_n = fx_n,\ n = 0, 1, 2, \dots \text{ and } \frac{\lambda}{1-\lambda} = r \tag{2.33}$$

Now for all n, we have

$$d(y_{n+1}, y_{n+2}) \leq rd(y_n, y_{n+1}) \leq \cdots \leq r^{n+1} d(y_0, y_1) \tag{2.34}$$

Now for any m > n,

$$\begin{aligned} d(y_m, y_n) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq (r^n + r^{n+1} + \cdots + r^{m-1}) d(y_0, y_1) \\ &\leq \frac{r^n}{1-r} d(y_0, y_1) \end{aligned} \tag{2.35}$$

Assume that $d(y_0, y_1) > 0$. Letting $n \to \infty$, (y_n) is a Cauchy sequence. Also, if $d(y_0, y_1) = 0$, then $d(y_m, y_n) = 0$ for all $m > n$ and hence (y_n) is a Cauchy sequence in X. Since $f(X)$ is complete, there exists $u, v \in X$ such that $y_n \to v = fu$. Now, we show that v is a common point of coincidence of S, T and f that is $v = fu = Su = Tu$.

$$\begin{aligned} d(fu, Su) &\leq d(fu, y_{2n+2}) + d(y_{2n+2}, Su) \\ &\leq d(v, y_{2n+2}) + d(Su, Tx_{2n+1}) \\ &\leq d(v, y_{2n+2}) + \lambda\,(d(fu, Su) + d(fx_{2n+1}, Tx_{2n+1})) \\ &\leq d(v, y_{2n+2}) + \lambda\,(d(fu, Su) + d(fx_{2n+1}, fx_{2n+2})) \\ &\leq d(v, y_{2n+2}) + \lambda\,(d(v, Su) + d(y_{2n+1}, y_{2n+2})) \end{aligned} \tag{2.36}$$

Letting n→ ∞, we have

$$d(v, Su) \leq \lambda\, d(v, Su) \tag{2.37}$$

Therefore, $d(v, Su) = 0$. Similarly,

$$\begin{aligned} d(Su, fu) &\leq d(Su, y_{2n+2}) + d(y_{2n+2}, fu) \\ &\leq d(Su, Tx_{2n+1}) + d(y_{2n+2}, v) \\ &\leq \lambda(d(fu, Su) + d(fx_{2n+1}, Tx_{2n+1})) + d(y_{2n+2}, v) \\ &\leq \lambda(d(v, Su) + d(y_{2n+1}, y_{2n+2})) + d(y_{2n+2}, v) \end{aligned} \tag{2.38}$$

Letting n→ ∞, we have

$$d(Su, v) \leq \lambda\, d(v, Su) \tag{2.39}$$

Hence, $|d(v, Su) - d(Su, v)| \leq 0$ and then $d(v, Su) = d(Su, v) = 0$. So $v = fu = Su$. Similarly, by using

$$d(fu, Tu) \leq d(fu, y_{2n+1}) + d(y_{2n+1}, Tu), \tag{2.40}$$

we can show that $fu = Tu$, it implies that v is a common point of coincidence of S, T and f. Now we show that f, S and T have unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$ for some u^* in X. First we show that $d(v, v) = d(v^*, v^*) = 0$. From condition (2.27), we have

$$\begin{aligned} d(v, v) &= d(Su, Tu) \\ &\leq \lambda(d(fu, Su) + d(fu, Tu)) \\ &\leq \lambda(d(v, v) + d(v, v)) = 2\lambda d(v, v) \end{aligned} \tag{2.41}$$

Since $0 \leq \lambda < \frac{1}{2}$, so the above inequality is possible if $d(v,v) = 0$. Similarly we can show that $d(v^*,v^*) = 0$.

Now we consider

$$d(v,v^*) = d(Su,Tu^*) \tag{2.42}$$
$$\leq \lambda(d(v,v) + d(v^*,v^*))$$

The above inequality is possible if $d(v,v^*) = 0$. which implies that $v = v^*$. If (S,f) and (T,f) are weakly compatible, by Lemma 2.13, S,T and f have a unique common fixed point.

Putting $S = T$ and $f = I$ an identity mapping in the above Theorem 2.16 yields Corollary 2.17.

Corollary 2.17 Let (X,d) be a complete dislocated metric space and the mappings $T,f : X \to X$ satisfy:

$$d(Tx,Ty) \leq \lambda(d(x,Tx) + d(y,Ty)) \tag{2.43}$$

for all $x,y \in X$ where $0 \leq \lambda < 1$. Then T has a unique fixed point.

Putting $S = T$ in the above Theorem 2.16 yields Corollary 2.18.

Corollary 2.18 Let (X,d) be a dislocated metric space and the mappings $S,T,f : X \to X$ satisfy:

$$d(Tx,Ty) \leq \lambda(d(fx,Tx) + d(fy,Ty)) \tag{2.44}$$

for all $x,y \in X$ where $0 \leq \lambda < 1$. If $T(X) \subseteq f(X)$ and f(X) is a complete subspace of X, then T and f have a unique point of coincidence. Moreover if (T,f) is weakly compatible, then T and f have a unique common fixed point.

Now, we present a fixed point theorem for mappings satisfying Geraghty-type contractive conditions in dislocated quasi-metric space.

Let $\mathcal{S}$ denotes the class of the real functions $\beta: \mathbb{R}_0^+ \to [0,1)$ satisfying the condition $\beta(t_n) \to 1$ implies $t_n \to 0$. An example of a function in $\mathcal{S}$ may be given by $\beta(t) = e^{-t}(t+1)^{-1}$ for $t > 0$ and $\beta(0) \in [0,1)$. Similarly the function $\beta: \mathbb{R}_0^+ \to [0,1)$ defined by $\beta(t) = e^{-t}$ for $t > 0$ and $\beta(0) \in [0,1)$, belongs to the class $\mathcal{S}$.

Observe that we do not assume that β is continuous in any sense. We only require that if β gets here one, it does so only near zero. In an attempt to generalize the Banach contraction principle, Michael A. Geraghty proved in 1973 the following.

Theorem 2.19 [17]Let $T: X \to X$ be a contraction on a complete metric space satisfying

$$d(Tx,Ty) \leq \alpha\big(d(x,y)\big)d(x,y), \tag{2.45}$$

where $\alpha \in \mathcal{S}$ .Then for any choice of initial point x_0, the iteration $x_n = Tx_{n-1}$ for $n > 0$, converges to the unique fixed point x_∞ of T in X.

We then have the following theorem.

Theorem 2.20 Let (X,d) be a complete dislocated quasi-metric space and $T: X \to X$ be a self map. Suppose that there exists $\beta \in \mathcal{S}$ such that

$$d(Tx,Ty) \leq \beta\big(d(x,y)\big)d(x,y). \tag{2.46}$$

holds for all $x,y \in X$ .Then T has a unique fixed point $x^* \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to x^* when $n \to \infty$.

Proof Let x_0 be arbitrary in X . Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T, and hence the proof is completed. Thus, throughout the proof, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (2.46), we get

$$0 < d(x_{n+1}\, x_{n+2}) = d(fx_n, fx_{n+1}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}). \tag{2.47}$$

Thus, we conclude that $d(x_{n+1}\, x_{n+2}) < d(x_n\, , x_{n+1})$ for all $n \in \mathbb{N}$. So, the sequence $\{d(x_n\, x_{n+1})\}$ is nonnegative, non-increasing and bounded from below. Hence, there exists $r \geq 0$ such that

$lim_{n\to\infty} d(x_n\, x_{n+1}) = r.$ We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, due to (2.47), we have

$$\frac{d(x_{n+1},x_{n+2})}{d(x_n,x_{n+1})} \leq \beta\big(d(x_n, x_{n+1})\big) < 1. \tag{2.48}$$

which yields that $lim_{n\to\infty} \beta\big(d(x_n, x_{n+1})\big) = 1$. By property of $\beta \in \mathcal{S}$, we derive that $lim_{n\to\infty} d(x_n\, x_{n+1}) = 0$. We shall show that (x_n) is a Cauchy sequence. Suppose, on the contrary, that (x_n) is not a Cauchy sequence. Thus, there exists $\epsilon > 0$ such that, for all $k > 0$, there exist $m(k) > n(k) > k$ with (the smallest number satisfying the condition below)

$$d\big(x_{m(k)}, x_{n(k)}\big) \geq \epsilon \text{ and } d\big(x_{m(k)-1}, x_{n(k)}\big) < \epsilon. \tag{2.49}$$

Then we have

$$\begin{aligned} \epsilon &\leq d\big(x_{m(k)}, x_{n(k)}\big) \\ &\leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)}\big) \\ &< d\big(x_{m(k)}, x_{m(k)-1}\big) + \epsilon. \end{aligned} \tag{2.50}$$

Letting $k \to \infty$ in the above inequality, we have

$$lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \epsilon \tag{2.51}$$

By using (2.50) and (2.51), we obtain

$$lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \epsilon \tag{2.52}$$

Thus from (2.46), we have

$$\begin{aligned} d\big(x_{m(k)}\, x_{n(k)}\big) &= d\big(Tx_{m(k)-1}, Tx_{n(k)-1}\big) \\ &\leq \beta\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) d\big(x_{m(k)-1}, x_{n(k)-1}\big) \\ &< d\big(x_{m(k)-1}, x_{n(k)-1}\big) \end{aligned} \tag{2.53}$$

Hence, we conclude that

$$\frac{d\big(x_{m(k)}\, x_{n(k)}\big)}{d\big(x_{m(k)-1},x_{n(k)-1}\big)} \leq \beta\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) < 1 \tag{2.54}$$

keeping (2.53) and (2.54) in mind and letting $n \to \infty$ in the above inequality, we derive that $lim_{k\to\infty} \beta\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) = 1$ and so, $lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = 0$. Hence, $\epsilon = 0$, which is a contradiction. So, we conclude that $\{x_n\}$ is a Cauchy sequence. But X is a complete dq-metric space, it follows that there exists $x^* \in X$ such that $\{x_n\}$ converges to x^* i.e. $lim_{n\to\infty} d(x_n\, , x^*) = lim_{n\to\infty} d(x^*, x_n) = 0$. Now we will prove that x^* is a fixed point of T. We have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(\mathrm{x}_{n+1}\, , Tx^*) \\ &\leq d(x^*, x_{n+1}) + \beta(d(x_n\, , x^*))d(x_n\, , x^*) \\ &\to 0 \;\; as\; n \to \infty. \end{aligned} \tag{2.55}$$

This implies that $d(x^*, Tx^*) = 0$. Similarly, we can show that $d(Tx^*, x^*) = 0$. Hence $Tx^* = x^*$. Finally to prove that last part of the theorem; let us assume that $Tx^* = x^*$, $Ty^* = y^*, x^* \neq y^*, x^*, y^* \in X$. Therefore,

$$\begin{aligned} 0 < d(x^*, \mathrm{y}^*) &= \mathrm{d}(\mathrm{Tx}^*\, , \mathrm{Ty}^*) \\ &\leq \beta\big(\mathrm{d}(\mathrm{x}^*, \mathrm{y}^*)\big)\mathrm{d}(\mathrm{x}^*, \mathrm{y}^*) < d(\mathrm{x}^*, \mathrm{y}^*) \end{aligned} \tag{2.56}$$

This is a contradiction. Hence fixed point x^* of T is unique in X .

Now, we present one example to illustrate above Theorem 2.20.

Example 2.21 Let $X = [0,1]$ with complete d_q -metric defined by $d(x,y) = \max\{x,y\}$ .The function $\beta: \mathbb{R}_0^+ \to [0,1)$ defined by $\beta(t) = \frac{e^{-t}}{t+1}$, for $t > 0$ and $\beta(0) \in [0,1)$. Then $\beta \in \mathcal{S}$. Now we define a self-map $T: X \to X$ by $T(x) = \frac{x}{6}$. Take $x = 1, y = 0$ and obtain that

$$d(T1, T0) = d\left(\frac{1}{6}, 0\right) = \frac{1}{6},$$

$$\beta\big(d(1,0)\big)d(1,0) = \beta(1).1 = \beta(1).1 = \frac{e^{-1}}{1+1} = \frac{1}{2e} > \frac{1}{6}$$

On the other hand, take $x, y \in X$ with $x \geq y$. Then

$$d(Tx, Ty) = d\left(\frac{x}{6}, \frac{y}{6}\right) = \frac{x}{6},$$

$$\beta\big(d(x,y)\big)d(x,y) = \beta(x).x = \frac{xe^{-x}}{x+1} \geq \frac{1}{2e} > \frac{1}{6}, \forall\, x \in [0,1].$$

Hence T satisfies condition of Theorem 2.20 and T has a unique fixed point $x^* = 0$.

3. CONCLUSION

To summarize, we have explored the properties of dislocated quasi-metric spaces and dislocated metric spaces. Also discuss the difference between metric space and generalizations of metric space. We established a fixed point theorem for a self-mapping in complete dislocated metric spaces under contractive conditions related to a decreasing map $\tau : \mathbb{R}_0^+ \to [0, 2^{-1})$. We obtained sufficient conditions for existence of points of coincidence and common fixed points of three self mappings satisfying a contractive type conditions in dislocated metric spaces. Also we presented a fixed point theorem for mappings satisfying Geraghty-type contractive conditions in dislocated quasi-metric space. We also present some examples in support of our results.

Competing Interests

No conflict of interest was declared by the authors.

Author's Contribution

Both authors contributed equally and significantly to writing this paper. Both authors read and approved the final manuscript.

Acknowledgements

The authors are grateful to the reviewers and the editor for their useful comments and suggestions.

REFERENCES

[1]. C. T. Aage and J. N. Salunke, Some results of fixed point theorem in dislocated quasi-metric space, Bulletin of Marathadawa Mathematical Society, 9(2008), 1-5.

[2]. C. T. Aage, J. N. Salunke, The Results on Fixed points in Dislocated and Dislocated quasi-metric space, Applied Mathematical Sciences, 2, 2941–2948 (2008).

[3]. P. Hitzler, Generalized metrics and Topology in Logic Programming Semantics, Ph.D Thesis, National Univeristy of Ireland, University College Cork, (2001).

[4]. A. Isufati, Fixed point theorem in dislocated quasi-metric spaces, Applied Mathematical Sciences, 4(2010), 217-223.

[5]. A. Muraliraj et al., Generalized Fixed Point Theorems in Dislocated Quasi Metric Spaces, Advance in Inequalities and Applications, ISSN 2050-7461, 2014, 2014:17.

[6]. F. M. Zeyada, G. H. Hassan, M. A. Ahmed, A Generalization of a Fixed point theorem due to Hitzler and Seda in Dislocated Quasi-Metric Spaces., The Arabian Journal for Science and Engineering, 31, 111–114 (2005).

[7]. A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, Fixed Point Theory Appl., 2012 (2012), 10 pages.

[8]. S. Bennani, H. Bourijal, S. Mhanna, D. El Moutawakil, Some common fixed point theorems in dislocated metric spaces, J. Nonlinear Sci. Appl., 8 (2015), 86-92.

[9]. P. Hitzler and A. K. Seda, Dislocated Topologies, J. Electr. Engin., 51(2000), 3-7.

[10]. D. Panthi, K. Jha, A common fixed point of weakly compatible mappings in dislocated metric space, Kathmandu University J. Sci., Engineering and Technology, 8 (2012), 25-30.

[11]. B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expression, Indian Journal of Pure and Applied Mathematics, 6(1975), 1455-1458.

[12]. B. E. Rohades, A comparison of various definitions of contractive mappings, Transfer. Amer. Soc., 226(1977), 257-290.

[13]. Mujeeb Ur Rahman and M. Sarwar, Fixed point results in dislocated quasi-metric spaces, International Mathematical Forum, 9(2014), 677-682.

[14]. M. Sarwar, Mujeeb Ur Rahman and G. Ali , Some fixed point results in dislocated quasi-metric (dq-metric) Spaces, Journal of Inequalities and Applications, 2014, 278:2014, 1-11.

[15]. A. S. Saluja, A. K. Dhakda and D. Magarde, Some fixed point theorems for expansive type mapping in dislocated metric space, Mathematical Theorey and Modeling, 3(2013).

[16]. R. D. Daheriya, Rashmi Jain, and Manoj Ughade “Some Fixed Point Theorem for Expansive Type Mapping in Dislocated Metric Space”, ISRN Mathematical Analysis, Volume 2012, Article ID 376832, 5 pages, doi:10.5402/2012/376832.

[17]. Geraghty, M: On contractive mappings. Proc. Am. Math. Soc. 40, 604-608 (1973).

[18]. G. Jungck and B.E. Rhoades, Fixed Points For Set Valued Functions without Continuity, Indian J. Pure Appl. Math., **29** (3)(1998), 227-238.

[19]. S.G. Mathews, “Metric Domains for Completeness”, Ph.D. Thesis. Research Report 76, Dept. of computer science, University of Warwick, UK, 1986, pp. 1-127.

[20]. P. Finsler, Ueber Kurven und Flächen in allgemeinen Räumen. Leemann & Co., 1918.

[21]. W. Wilson, “On quasi-metric spaces”, American Journal of Mathematics, 53(3):675–684, 1931.

[22]. R. Stoltenberg, “On quasi Metric Spaces”. Duke Math. Jour., 36, (1969).

[23]. Willard, S. “General Topology”, Addition-Wesley, Reading, MA, 1970.

[24]. P Sumati Kumari, “On dislocated quasi metric, Journal of Advanced Studies in Topology”, Vol.3, No.2, 66-76.

[25]. G. Jungck, “Commuting mappings and fixed points,” The American Mathematical Monthly, vol. 83, no.4, pp. 261-263, 1976.

[26]. G. Jungck, “Compatible mappings and common fixed points,” International Journal of Mathematics and Mathematical Sciences, vol. 9, no.4, pp. 771-779, 1986.

[27]. G. Jungck, “Fixed points for non-continuous non-self mappings on non-metric space,” Far East Journal of Mathematical Sciences, vol. 4, pp. 199-212, 1996.

[28]. P. S. Kumari, V. V. Kumar, I. Rambhadra Sharma, “Common fixed point theorems on weakly compatible maps on dislocated metric space,” Mathematics Sciences, vol.6, pp 1-5. 2012.

[29]. Yijie Ren, Junlei Li, and Yanrong Yu, Common Fixed Point Theorems for Nonlinear Contractive Mappings in Dislocated Metric Spaces, Abstract and Applied Analysis Volume 2013, Article ID 483059, 5 pages, http://dx.doi.org/10.1155/2013/483059.

[30]. Mujeeb Ur Rahman and Muhammad Sarwar, Fixed Point Theorems for Expanding Mappings in Dislocated Metric Space, Mathematical Sciences Letters, 4, No. 1, 69-73 (2015), http://dx.doi.org/10.12785/msl/040114.

[31]. Ramabhadra Sarma1, J. Madhusudana Rao2 and S. Sambasiva Rao, Fixed Point Theorems In Dislocated Quasi-Metric Spaces, Mathematical Sciences Letters, 3, No. 1, 49-52 (2014), http://dx.doi.org/10.12785/msl/030108