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Approximate Solutions of the Airy Equation with Numerical Support of MATLAB

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Abstract: *The ordinary differential equation named Airy equation is solved in two approximate forms; the first one using the WKB method (Wentzel, Kramers, and Brillouin) where a brief development of the method is made, identifying the results that are applied during the solution and, the second form, that includes a simplified exposition of the Saddle Point method for the evaluation of integrals of which the Steepest Descent method is part. The Fourier transform of the differential equation is used to reach Airy's integral whichis evaluated with the appointed method that exists in the second form of the solution. The idea here is to discuss the importance of managing approximate methods in the solution of differential equations and recreating an equation with broad applications in physics and engineering.*

Keywords: *Airy Equation, WKB Method, Fourier Transform, Saddle Point Method, Steepest Descent*

1. INTRODUCTION

The Airy equation is called in honor ofGeorge Biddell Airy (1801-1892) of UK, who was mathematicians andastronomer. He graduated from Trinity College, Cambridge, was aprofessor of mathematics at and astronomy of the observatory there.Are many achievements as work on planetary orbits, measuring the mean density of the Earth, a method of solution of two-dimensional problems in solid mechanics.

He conducted extensive research in the field of mathematical physics and applied mathematics to astronomical calculations: in the field of observational astronomy proposed the "Airy disk", the minimum apparent size of a star.In 1935, it was named in his honor "Airy" to a lunar crater. Airy Martian crater also bears his name. Airy functions take their name from their work on the Airy equation.

There are important jobs in the literature equation Airy, such book of Vallee Olivier and Soares Manuel [1]. This book contains the relation of the Airy function with special functions; for example, the Bessel function. To study the Schrödinger equation in relation to the Airy equation is advisable to consult [2] and [3]. We rely on the calculation that takes place in [4]. Finally, it is important to mention the work [5] on themathematics of rainbow.The correct explanation of these bands depends on the wave theory of light developed in the early nineteenth century, primarily by Auguste Fresnel. The right application of Fresnel's theory to rainbows is due to George Biddell Airy; this theory was superseded in turn by the electromagnetic theory of Maxwell. The description of the rainbow that Maxwell's theory produces is in some extreme cases quite different from the one produced by the wave theory. However, for practical purposes Airy's theory is sufficient. Airy's theory requires numerical evaluation of an improper integral named after him. How to carry out this evaluation also has an interesting history.

We reproduced the development of [4], but we added graphical analysis and we point some considered necessary omissions of the authors;this calculation is based on WKB method and the saddle point method that are briefly explained below

The WKB approximation or WKB method is a way for finding approximate solutions to linear differential equations with spatially varying coefficients. In quantum mechanics in which the wavefunctionis recast as an exponential function, and then either the amplitude or the phase is taken to be slowly changing. The name is aninitial for Wentzel–Kramers–Brillouin.Generally, WKB theory is a method for approximating the solution of a differential equation whose highest derivative is multiplied by a small parameter.

The method of steepest descent or stationary phase method or saddle-point method is an extension of Laplace's method for approximating an integral, where one deforms a contour integral in the complex plane to pass near a stationary point (saddle point), in roughly the direction of steepest descent or stationary phase. The saddle-point approximation is used with integrals in the complex plane, whereas Laplace’s method is used with real integrals. The method of steepest descent was first published by Debye (1909), who used it to estimate Bessel functions and pointed out that it occurred in the unpublished note Riemann (1863) about hypergeometric functions. The contour of steepest descent has a minimax property, described some other unpublished notes of Riemann, where he used this method to derive the Riemann-Siegel formula.

Next it has the display saddle point of

$$z = x^2 - y^2 \tag{1}$$

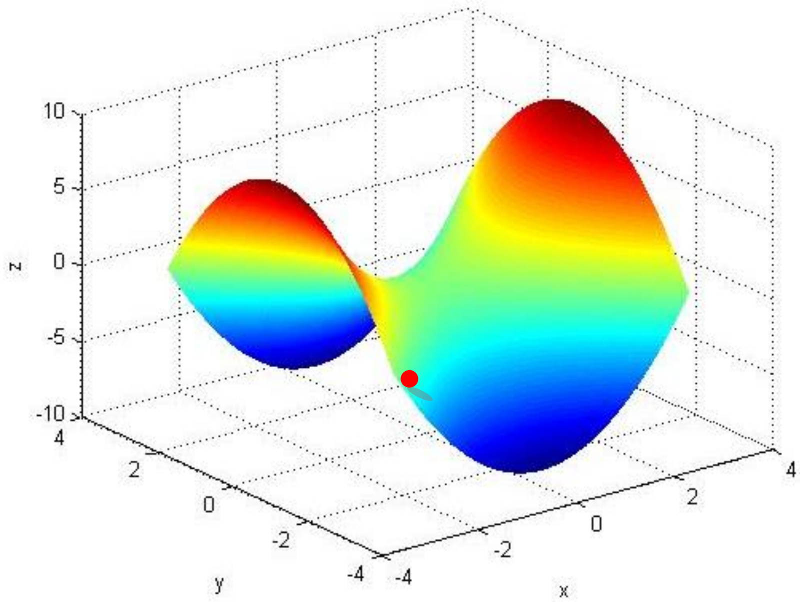

Fig. 1. *Plot of saddle point (red).*

Thisin addition to its application in science and engineering, the Airy equation proves interesting for a vast number of reasons. This expression is asecond ordinary differential equation of the form

$$\frac{d^2 y}{dx^2} = xy \tag{2}$$

Is commonly used in quantum physics quantum.

The Airy functions are the called $Ai(x)$ and $Bi(x)$, are solutions of (2) For value real of x the Airy function of first kind is defined

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)dt \tag{3}$$

Another solution is the Airy function of second kind

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[e^{\left(-\frac{t^3}{3}+xt\right)} + \sin\left(\frac{t^3}{3} + xt\right)\right]dt \tag{4}$$

On the other hand, it can be seen that solutions of

$$\frac{d^2y}{dx^2} = \omega^2 xy \tag{5}$$

$$\frac{d^2y}{dx^2} = xy - Ey \tag{6}$$

Are also Airy functions with rescaled or displaced arguments.Equation (6) is the Schrödinger equation of aparticle in a uniform external camp.

If we focus on its nature, it can be seen that for negative x the solutions are oscillatory, similar to the trigonometric functions, and for positive a solution grows exponentially while another one decreases in like manner. This work deals with equation (3),an example is shown in Figure 2:

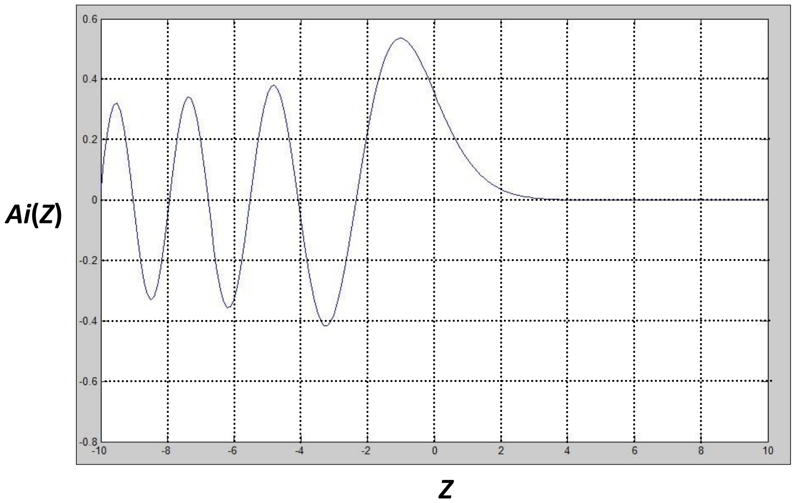

Fig. 2. *Graph of the Airy function for* $y(x) = Ai(z)$.

The other solution is displayed in figure 3.

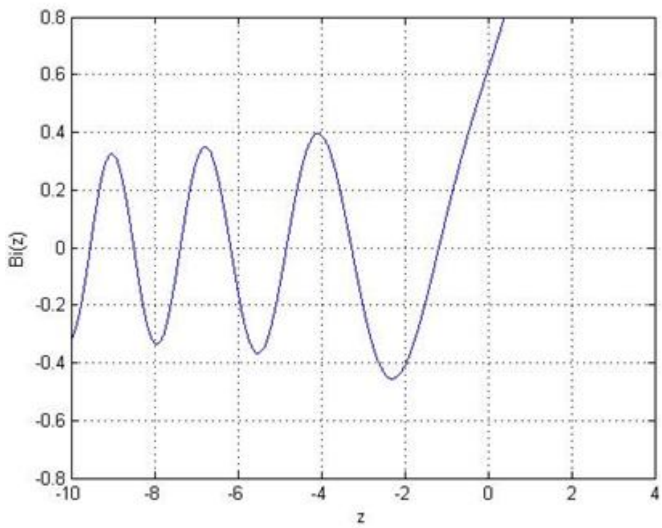

Fig. 3. *Graph of the Airy function for* $y(x) = Bi(z)$.

The numerical solution with MATLAB is in figure 4

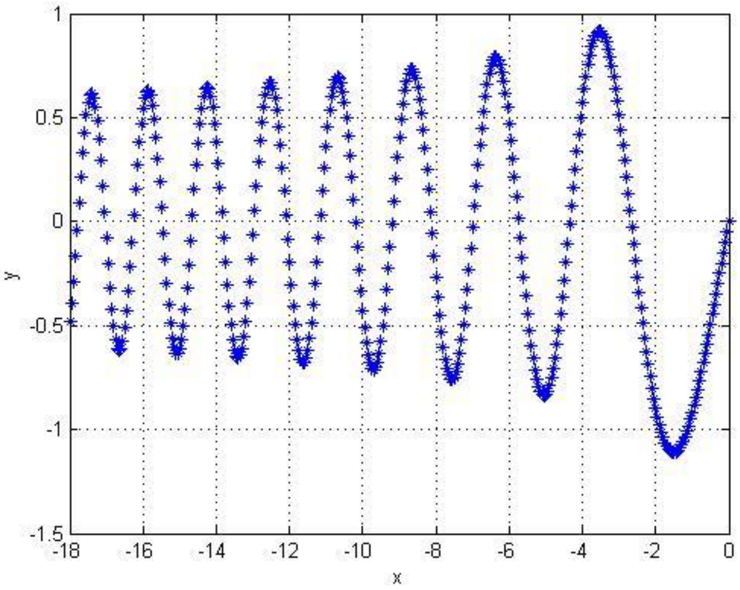

Fig. 4. *Graph of the numerical solution of* $Ai(x)$.

The approximate solution methods, more results of interest are obtained the following explains.

2. Solution using the WKB Method

We proceed from the fact that the WKB method provides solutions to equations of the following form

$$\frac{d^2 y}{dx} + f(x)y = 0 \tag{7}$$

Onlyif consists of smooth and slow variations.

The WKB method is developedfor applications in quantum mechanics. If we define a solution of the form

$$y = e^{i\phi(x)} \tag{8}$$

The differential equation (7) is determined by

$$-(\phi')^2 + i\phi'' + f = 0 \tag{9}$$

If we assume a small ϕ , a first approximation will be

$$\phi'(x) = \pm\sqrt{f(x)} \;\; \phi(x) = \pm\int \sqrt{f(x)}dx \tag{10}$$

The validity condition will be

$$|\phi''| \approx \frac{1}{2}\left|\frac{f'}{\sqrt{f}}\right| << |f| \tag{11}$$

A second approximation is easy

$$|\phi''| \approx \pm\frac{1}{2} f^{-1/2} f' \tag{12}$$

Substituting in (9)

$$\phi' = \sqrt{\tfrac{i}{2} f^{-1/2} f' + f} \tag{13}$$

When doing the expansion of radical

$$\phi' = \pm\sqrt{f} + \frac{i}{4}\frac{df}{f} \tag{14}$$

Then

$$\phi(x) \approx \pm\int \sqrt{f(x)}dx + \frac{i}{4}\ln f \tag{15}$$

Then

$$y = e^{i\phi(x)} = e^{i\int\sqrt{f(x)}dx + \frac{i}{4}\ln f} \tag{16}$$

Selecting both signs in a solution as a linear combination

$$y(x) \approx \frac{1}{(f(x))^{1/4}}\left\{c_+ e^{i\int\sqrt{f(x)}dx} + c_- e^{-i\int\sqrt{f(x)}dx}\right\} \tag{17}$$

The WKB function associated to (7) is:

$$W_\pm = [f(x)]^{-1/4}\exp\left[\pm\int_{x_0}^{x}\sqrt{f(x)}dx\right] \tag{18}$$

Likewise, it is possible to determine that the WKB function associated to (6) meets

$$W_\pm' = f^{1/4} e^{\int\sqrt{f}dx} - \frac{1}{4} f^{-5/4} f' e^{\int\sqrt{f}dx} \tag{19}$$

$$W_\pm'' = f(x) + \frac{1}{4}\frac{f''}{f} - \frac{5}{16}\left(\frac{f'}{f}\right)^2 \tag{20}$$

$$W_\pm'' + [f(x) + \frac{1}{4}\frac{f''}{f} - \frac{5}{16}\left(\frac{f'}{f}\right)^2]W_\pm = 0 \tag{21}$$

If we define

$$g(x) = \frac{1}{4}\frac{f''}{f} - \frac{5}{16}\left(\frac{f'}{f}\right)^2 \tag{22}$$

Then $W_\pm$ is the exact solution of

$$W_\pm'' + [f(x) + g(x)]W_\pm = 0 \tag{23}$$

and an approximate solution of (7). Then

$$y(x) = \alpha_+(x)W_+(x) + \alpha_-(x)W_-(x) \tag{24}$$

is proposed as the linear combination of the WKB functions

Where the $\alpha_{\pm}$ can be obtained for the Airy equation

$$y'' + xy = 0 \tag{25}$$

Then

$$f(x) = x \quad g(x) = -\frac{5}{16}x^{-2} \tag{26}$$

Likewise, it is possible to determine that

$$\alpha_{\pm} = \frac{A}{2}e^{\pm i\delta} \tag{27}$$

Then using (16) and (26) in equation (24)

$$y(x) \approx \frac{1}{x^{1/4}}\left\{c_{+}e^{i\frac{2}{3}x^{3/2}} + c_{-}e^{-i\frac{2}{3}x^{3/2}}\right\}\alpha_{\pm} \tag{28}$$

and the WKB approximation of the Airy equation for $x \to \infty$ is

$$y \approx Ax^{-1/4}\cos\left(\frac{2}{3}x^{3/2} + \delta\right) \tag{29}$$

Theconstant A and δ are defined with the next method

3. Saddle Point Method

This method is important in the approximate evaluation of integrals. In general, the Steepest Descent method is applicable to integrals of the form

$$I(\alpha) = \int_{C} e^{\alpha f(z)}dz \tag{30}$$

(for α large and positive)

where is a road in the complex plane such that, at the end of the road, it does not contribute to the integral.If

$$f(z) = u + iv \tag{31}$$

the method attempts to compress the contour as much as possible. For example if

$\frac{\partial^2 u}{\partial x^2} < 0$ then $\frac{\partial^2 u}{\partial y^2} > 0$ there will be a flat section on the surface $u(x, y)$ where

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ proves to be a saddle point z_0. Very close to it, we have that

$$f(z) \approx f(z_0) + \frac{1}{2}f''(z_0)(z - z_0)^2 \tag{30}$$

Let the equation be

$$f''(z_0) = \rho e^{i\theta} \quad , \quad z - z_0 = se^{i\phi} \tag{31}$$

Then,when substituting in (30),

$$\begin{aligned} u &\approx u(x_0, y_0) + \frac{1}{2}\rho s^2 \cos(\theta + 2\phi) \\ v &\approx v(x_0, y_0) + \frac{1}{2}\rho s^2 sen\,(\theta + 2\phi) \end{aligned} \tag{32}$$

Using this approximation in the integral (30)

$$I(\alpha) \approx e^{\alpha f(z_0)} \int_{-\infty}^{\infty} e^{-\left(\alpha/2\right)\rho s^2} e^{i\phi} ds \approx \sqrt{\frac{2\pi}{\alpha\rho}} e^{\alpha f(z_0)} e^{i\phi} \tag{33}$$

where ϕ has a value of $-\theta/2 \pm \pi/2$ depending on the direction of the travel to the saddle point.

Now, the asymptotic calculation for large is briefly shown. We move from the Fourier transform of (7).Here we have used the following

$$\frac{dy}{dx} = -\omega^2 g(\omega) \tag{34}$$

$$xy = i\frac{dg}{d\omega} \tag{35}$$

$$-\omega^2 g(\omega) + i\frac{dg}{d\omega} = 0 \tag{36}$$

$$g(\omega) = \int_{-\infty}^{\infty} y(x) e^{i\omega x} dx \tag{37}$$

$$y(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty} g(\omega) e^{i\omega x} dx \tag{38}$$

Let the solution of (36)

$$g(\omega) = e^{\frac{1}{3}\omega^3} \tag{39}$$

Therefore, when reversing the transform we have:

$$y(x) = A\int \frac{d\omega}{2\pi} e^{i\left[\omega x - \frac{\omega^3}{3}\right]} \tag{40}$$

which is the so-called Airy integral. Since it is not trivial, the solution using the Saddle Point method is proposed.Hence we have that:

$$\left(\omega x - \frac{\omega^3}{3}\right) = x\left(\omega - \frac{\omega^3}{3x}\right) \tag{41}$$

$$f(\omega) = i\left(\omega - \frac{\omega^3}{3x}\right) \tag{42}$$

$$f'(\omega) = i\left(1 - \frac{\omega^2}{x}\right) \tag{43}$$

The extreme is

$$f'(\omega_0) = 0 \Rightarrow \omega_0 = \pm\sqrt{x} \tag{44}$$

Likewise,

$$f''(\omega_0) = -\frac{2i\omega_0}{x} \tag{45}$$

Hence, by using equations (31) and (44) we have

$$\rho e^{i\theta} = \mp\frac{2i}{\sqrt{x}} \quad \rho = \frac{2}{\sqrt{x}} \quad \theta = \mp\frac{\pi}{2} \quad \phi = \mp\frac{\pi}{4} \tag{46}$$

Angles defined like this are conditions of the saddle point, then, substituting (44) in (42) we have

$$f(\omega_0) = \pm\frac{2}{3}i\sqrt{x} \tag{47}$$

Therefore,we can build the solution by using (46) and (47) in (33)

$$y(x) \approx \sum_{\pm} \sqrt{\frac{2\pi\sqrt{x}}{2x}} \exp\left[\pm\frac{2}{3}ix^{3/2}\right] \exp\left[\mp\frac{i\pi}{4}\right] \tag{48}$$

$$y(x) \approx \frac{2\sqrt{\pi}}{x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \tag{49}$$

In this expression there are values for A and δ from the equation (29).In Figure 5 the solution is plotted.

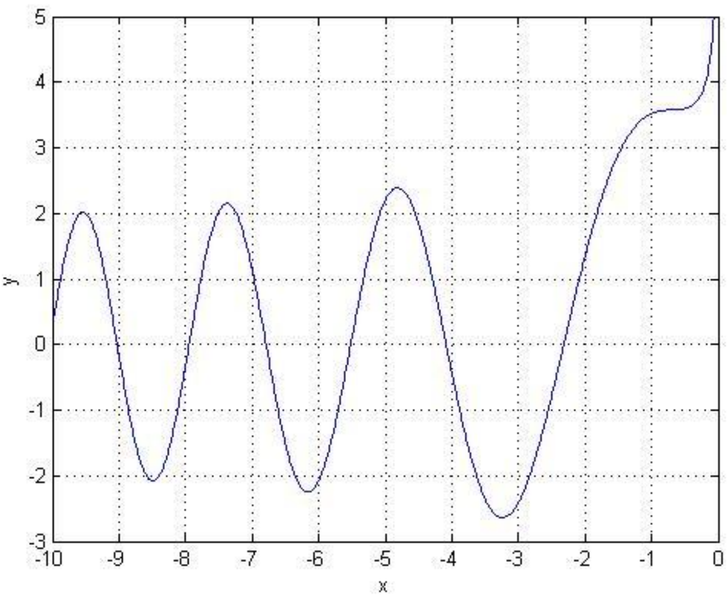

Fig.5. *Solution of saddle point method $y(x)$*

For last, the 3D display of the equations (3) and (4) done by MATHEMATICA software are presented.

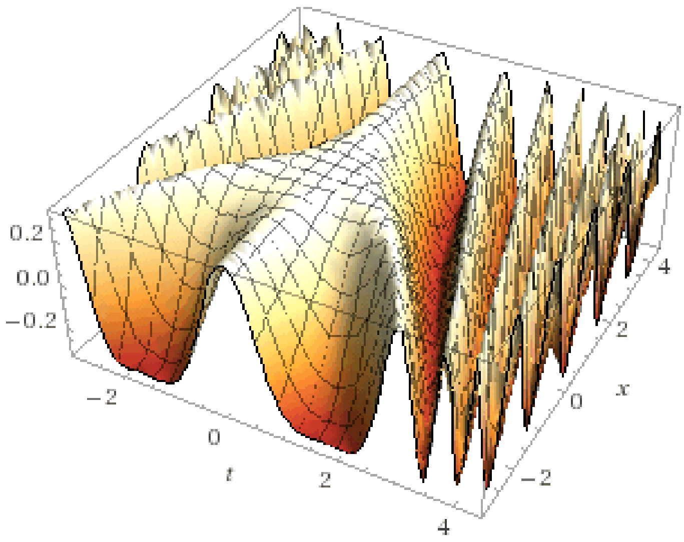

Fig.6. *Airy function of first kind,equation (3)*

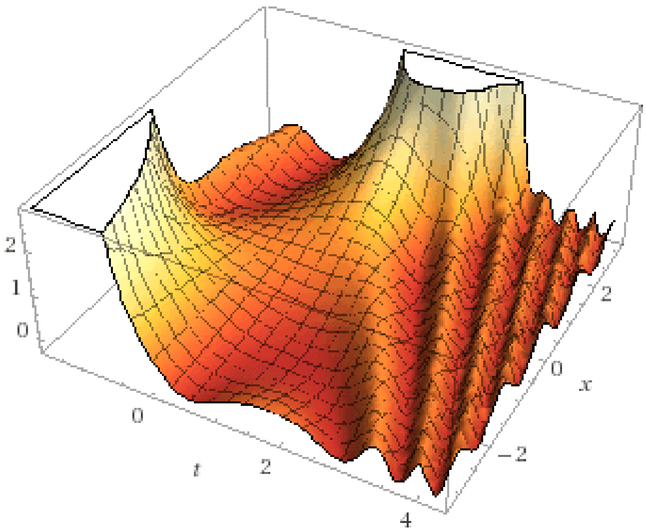

Fig.7. *Airy function of second kind,equation (4)*

4. Results and Discussion

In the introduction, it is proposed to present the graph of each type of Airy function given, (Figures2 and 3), to figure out a better idea of the solution. Also, we consider as a didactic value plotting the saddle point (Figure 1) since the most of the timethis detail is omitted in the educational materials. The numerical solution of Figure 4 is easily obtained with MATLAB. Then in the WKB calculation [4] we believe it is necessary to point out the equations (13) and (14), which we consider essential to specify in the algebraic development. In the saddle point method, we added the definition of the Fourier transform (Equation (34) and (35)) because we believe these equations should be clarified in the calculation. Similarly, it is proposed to supplement the mathematical development with the inclusion of the equation (38) and (39) which greatly helps to understand the method. The solution of saddle point is plotted in Figure 5 in order tovalidate its conformation to the Airy function.For last,the 3D graphics of Figures 6 and 7 are done by the midpoint integrator from MATHEMATICA software.

5. Conclusion

In this work, two forms for using approximation methods were shown. We believe that discussing these types of problems has educational value for students of science and engineering. Likewise, it should be noted that it is possible to express the Airy function in terms of the Bessel and hypergeometric functions and these generate very interesting results in the study of special functions. Another aspect of this topic is the search for other types of solutions in terms of power series, which, as a whole, define the mathematical wealth of the Airy equation.

REFERENCES

[1]. O. Vallee and M. Soares, *Airy functions and applications to physics.* ed. Imperial College Press, pp. 5-7.

[2]. L. De la peña and M. Villavicencio,*Problemas de Mecánica Cuántica*. Edit. FCE, México 2003.

[3]. C. Trallero and M. Leyva,*Problemas de Mecánica Cuántica*. Edit. Pueblo y Educación. Cuba, 1989.

[4]. J. Mathews and R. Walter,*Mathematical Methods of Physics*. Edit. Addison Wesley EEUU 1964.

[5]. B. Casselman., The mathematics of Rainbow. Monthly essays on mathematical topics. American Mathematical Society

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