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Max- Flow Min-Cut and Konig's Theorem's Using the Uni-Modular Matrices and Concurrent Multi-Commodity Flow (CMFP) in Linear Programming

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Abstract: *In this paper, we use a different proof technique to prove the Konig's theorem and Max-flow Min-cut theorem. We consider the linear programming formulation of the problem and show that the optimal values of primal and dual are equal. We use the total unimodularity property of coefficient matrix and the fundamental theorem of duality in linear program to drive this equivalence. The total unimodularity of the coefficient matrix helps in determining the integrality of the solution. we present the proof of the Max-flow Min-cut theorem and Konig's theorem using the properties of total uni-modular matrices in linear programming. We discuss the problem of Concurrent Multi-commodity Flow (CMFP) and present a linear programming formulation.*

Keywords: *Unimodular matrix, Maximum flow, Concurrent Multi-commodity Flow*

1. Introduction

The Multi-commodity flow problem is a more generalized network flow problem. The problem of finding a maximum flow in a multi-commodity network arises in many network instances. In a Multi-commodity flow problem, there exists $k \geq 1$ commodities each having its own source and sink. And we give an introduction to the Concurrent Multi-commodity Flow problem (CMFP) [15]and present the linear programming formulation for the problem and its dual. In CMFP, every commodity is assigned a demand D_i , our objective is to assign flows to the commodities so as to maximize a fraction λ such that the flow for any commodity is at least λD_i units.

1.1. Preliminaries

Definition 1.1.1 (Unimodular Matrix) A matrix M over real numbers is said to be unimodular if every square sub matrix of M has determinant equal to 0,1 or -1.

Examples of totally uni-modular matrices,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Definition1.1.2. (Linear Program) Let P be a maximization problem. Consider this as the primal. Then the linear program formulation can be given as Maximize $C^T X$.

$$\text{Subject to } Ax \le b. \tag{1.1.1}$$

Where A is a mXn matrix and c, x, b are column vectors of order n, n, m respectively. C^T denotes the transpose of C.

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Definition 1.1.3.(Dual of a Linear Program) The dual of the LPP(P), say D(P),can be given as

$$\text{Minimize } b^T y \text{ .,}$$

$$\text{Subject to } A^T y \ge c \text{ . } y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix}$$

$y \ge 0$; Where y is a column vector of order m.

Theorem1. (Weak Duality) For any feasible solutions $x^/$ and $y^/$ of P and D(P), $c^T x^/$ is always less than or equal to $b^T y^/$.

Proof. For a feasible $x^/$ and $y^/$ of P and D(P), the inequality constraints will be satisfied. Consider the inequalities in P and multiply with $y^{/T}$ on both sides.

$Ax^/ \le b$

$$y^{/T} Ax^/ \le y^{/T} b \tag{1.1.2}$$

Now, Consider the objective function $c^T x^/$ of P. We have,

$$A^T y^/ \ge c \qquad \Rightarrow (A^T y^/)^T x^/ \ge c^T y^/ \ \Rightarrow y^{/T} Ax^/ \ge c^T x^/ \tag{1.1.3}$$

Now from equations 1.1.2 and 1.1.3 we have

$$c^T x^/ \le y^{/T} Ax^/ \le y^{/T} b \tag{1.1.4}$$

We can see that $y^{/T} b = b^T y^/$. Since,

$$\begin{matrix} y_1 & y_2 & . & . & . & y_m \end{matrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix} = b_1 y_1 + b_2 y_2 + b_3 y_3 + ... + b_m y_m \tag{1.1.5}$$

$$= \begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_m \end{bmatrix} = b^T y^{/}$$

Therefore, $c^T x^{/} \le y^{/^T} A x^{/} \le b^T y^{/}$ (1.1.6)

Theorem2: (Strong Duality) If the primal P has an optimal solution, x^*, then the dual D(P) also has an optimal solution, y^*, such that $c^T x^* = b^T y^*$.

Theorem3. (Unimodular LP) For the LPP(P), if A is a unimodular matrix and b is integral then some optimal solution is integral.

1.2. Konig's Theorem

Theorem4. (Konig's theorem, [1931]) In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

Proof. Let $G(V,E)$ be a bipartite graph and X,Y be the two partitions of V. Let $|X| = m, |Y| = n$ and $\{1,2,3,...,n\} \in X, \{m+1, m+2, ..., m+n\} \in Y$.

If E is empty, then the size of both the minimum vertex cover and maximum matching is zero. Without loss of generality we shall assume that E is non empty and $|E| = r$. We shall introduce two variables p and q corresponding to every vertex and edge respectively. The linear program (P) for finding the maximum matching is as follows,

$$\text{Maximize} \sum_{(i,j)\in E} q_{ij} \tag{1.2.1a}$$

$$\text{Subject to} \sum_{j:(i,j)\in E} q_{ij} \le 1 \qquad \forall i \in X \tag{1.2.1b}$$

$$\sum_{i:(i,j)\in E} q_{ij} \le 1 \qquad \forall j \in Y \tag{1.2.1c}$$

$$q_{ij} \in \{0,1\} \ \forall (i,j) \in E \tag{1.2.1d}$$

The first two constraints imply that at most one edge can be selected corresponding to every node. Here, the given problem is an Integer Linear Program. We relax the integrality constraints on q_{ij} so that it can take on decimal values. The resulting LPP, say $P^{/}$, is

$$\begin{aligned} &\text{Maximize} \sum_{(i,j)\in E} q_{ij} \\ &\text{Subject to} \sum_{j:(i,j)\in E} q_{ij} \le 1 \qquad \forall i \in X \\ &\sum_{i:(i,j)\in E} q_{ij} \le 1 \qquad \forall j \in Y \\ &q_{ij} \ge 0 \ \forall (i,j) \in E \end{aligned} \tag{1.2.2}$$

Let A be the coefficient matrix of the LPP ($P^{/}$). By Theorem 1.2.1 below,

A is unimodular. Hence by Lemma , there exists an optimal integral solution to $P^{/}$. Obviously, every element of the solution is either 0 or 1. Otherwise it would violate the constraints. Hence optimal solution of $P^{/}$ is also optimal solution of P. This is the size of the maximum matching in G.

Consider the LP in equation 1.2.2. We shall derive the dual of this LP formulation. Let p_i and p_j where $i \in X$, $j \in Y$ be the dual variables corresponding to the first and second set of constraints, respectively. The matrices corresponding to the primal formulation 1.1.1 and dual formulation 1.1.3 problems are as follows,

$$x = \begin{bmatrix} q_{ij} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{m+1} \\ \vdots \\ p_{m+n} \end{bmatrix} \tag{1.2.3}$$

Where the order of the matrices are, x is $rX1$, c is $rX1$, b is $(m+n)X1$, and y is $(m+n)X1$. The optimization function with respect to the definition 1.1.3 can be given as,

$$b^T y = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} X \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{m+1} \\ \vdots \\ p_{m+n} \end{bmatrix} \tag{1.2.4}$$

From Theorem 3.2.1, we know the coefficient matrix is unimodular. In the matrix A, with respect to a variable q_{ij}, every column has exactly two ones corresponding to i^{th} and j^{th} rows. Thus, in A^T every row has two ones, one in the i^{th} column and other in the j^{th} column. So, the set of constraints, $A^T y \geq c$, for the dual can be given as follows.

$$p_i + p_j \geq 1 \qquad \forall (i, j) \in E \tag{1.2.5}$$

Along with the non-negativity constraints, the dual can be given as,

$$\begin{aligned} &\text{Minimize } \sum_{i \in X} p_i + \sum_{j \in Y} p_j \\ &\text{Subject to, } p_i + p_j \geq 1 \qquad \forall (i, j) \in E \\ &p_i \geq 0 \qquad \forall i \in X \\ &p_j \geq 0 \qquad \forall j \in Y \end{aligned} \tag{1.2.6}$$

This actually is the LP formulation for the relaxed minimum vertex cover problem that we see below.

Now, consider the problem for finding the minimum vertex cover in G. Then the LPP formulation, say Q, is given as follows,

$$\text{Minimize } \sum_{i \in X} p_i + \sum_{j \in Y} p_j \tag{1.2.7}$$

Subject to, $p_i + p_j \geq 1 \qquad \forall (i, j) \in E$

$$p_i \in \ 0,1 \qquad \forall i \in X$$

$$p_j \in \ 0,1 \qquad \forall j \in Y$$

The above given problem is an ILP (Integer Linear Program). We shall relax the integrality constraints on x_i, x_j in order to obtain a LP. Let $Q^/$ be the new LP.

The new LP is as follows,

$$\text{Minimize } \sum_{i\in X} p_i + \sum_{j\in Y} p_j$$

$$\text{Subject to, } \quad p_i + p_j \geq 1 \qquad \forall (i, j) \in E \tag{1.2.8}$$

$$p_i \geq 0 \qquad \forall i \in X$$

$$p_j \geq 0 \qquad \forall j \in Y$$

Let B be the coefficient matrix of $Q^/$. It is easy to see that B is the transpose of A. Since A is unimodular, B is also unimodular. Therefore, by Theorem 1.2.1, $Q^/$ has an optimal integral solution in which all p_i and q_j are either 0 or 1. Hence, optimal solution to $Q^/$ is also optimal solution to Q. That is equal to the size of the minimum vertex cover.Now, by the Theorem 6, both the problems have same optimal solution.

Lemma1.2.1. The coefficient matrix A of the LPP($P^/$) is totally unimodular.

Proof. Clearly, A has m + n rows and r columns in which first m equations con-tribute the first m rows and second n equations contribute the remaining n rows. Each column of A has exactly two 1's, one in the first m rows and one in the last n rows. All other elements are 0. Let D be a square sub-matrix of order k. We will prove the theorem by using induction on k.

Clearly, for k = 1, $|D|$ = 0 or 1. Assume that all square sub-matrices of order $k-1$ have determinant equal to 0, 1 or -1. We shall consider different cases,

(1) If D has at least one column containing only zeros, then $|D|$ = 0.

(2) If D has at least one column containing only one one. Then $|D| = \pm|E|$ where E is the sub-matrix obtained from D by deleting the corresponding Column and the row containing the one. By induction $|E|$ = 0, 1 or -1. Hence $|D|$ = 0; 1 or -1.

(3) If every column of D has exactly two 1. In this case, the first 1 comes from the first m rows of A and the second 1 comes from the later n rows of A. Strictly, every column has exactly a single 1 within the rows that belong to the first m rows of A. So, the by performing row addition on all these rows, we obtain a row with all ones. The same argument applies for the remaining rows of D that came from the last n rows of A. Hence the rows of D are linearly dependent and $|D|$ =*0*. Hence A is unimodular.

1.3. The Max-Flow Min-Cut Theorem

Theorem5. (Ford-Fulkerson, 1956) In a Network G, let f be any maximum flow in G, then $\exists$ a cut (A, B) for which f(A, B) = c(A,B)

Proof. Let N(V, E, c, s, t) be the network with $|V|$ = n and $|E|$ = m. We shall write the linear programming formulation for the maximum flow.

We want to find the maximal flow that can be sent from the source vertex s to the sink vertex t. Let v be the value of any flow from s to t and x_{ij} be the flow sent along the arc (i,j). Let the vertices be labeled using integers 1 to n such that the source s is labeled as 1 and the sink t is labeled with n. Then the LPP(R) corresponding to the maximum flow is,

Maxmize v

Subject to

$$\begin{aligned} \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} - v &= 0 \qquad if \quad i = 1 \\ \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} + v &= 0 \qquad if \quad i = n \\ \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} &= 0 \qquad if \quad i = 2,3,\ldots,n-1 \\ x_{ij} &\le c_{ij} \qquad \forall (i,j)\in E \\ x_{ij} &\ge 0 \qquad \forall (i,j)\in E \\ v &\ge 0 \end{aligned} \tag{1.3.1}$$

The first constraints imply that the net flow out of the source vertex 1 is equal to v and the second constraints imply that the net flow into the sink vertex n is v. The third constraints imply that the total flow into any intermediate vertex is equal to the total out flow of that vertex. By relaxing the equality in these three constraints, we will obtain the following inequalities.

$$\begin{aligned} \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} - v &\ge 0 \qquad if \quad i = 1 \\ \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} + v &\le 0 \qquad if \quad i = n \\ \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} &\le 0 \qquad if \quad i = 2,3,\ldots,n-1 \end{aligned} \tag{1.3.2}$$

Let $R^{/}$ be the resulting LPP. In any optimal solution of $R^{/}$, the above three inequalities should satisfy with equality. Otherwise, by adding all the LHS and RHS we get 0 < 0, a contradiction. Since, on the LHS for every arc (i, j), x_{ij} is added once and subtracted once, so the sum will result in a zero. So, they will satisfy with equality. Therefore, the optimal solution of $R^{/}$ will also be the optimal solution of R.

Consider the LP for Max-flow in equation 1.3.1. If we try to convert this into matrix form, the corresponding matrices will be,

$$x = \begin{bmatrix} x_{ij} \\ \vdots \\ \vdots \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_{ij} \\ \vdots \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Now, we shall derive the dual for the above LP in equation. Let u_i where $1 \le i \le n$ be the dual variables corresponding to the first three set of equations (flow constraints) and y_{ij} where $(i, j) \in E,$ be the dual variables corresponding to the fourth set of constraints (capacity constraints),Then the matrix y is,

$$y = \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ y_{ij} \\ \vdots \end{bmatrix} \tag{1.3.4}$$

Therefore the objective function for the dual can be given as,

$$b^T y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_{ij} \\ \vdots \end{bmatrix} \geq \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ y_{ij} \\ \vdots \end{bmatrix} \tag{1.3.5}$$

Consider the coefficient matrix, say A, of the LP formulation for Max-flow. We shall see the properties of this matrix. There will be (n+m) rows corresponding to $m\,x_{ij}$ the (n+m) constraints and m+1 columns corresponding to variables. The coefficient matrix A, will look as below,

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ x_{ij} \\ \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \vdots\vdots\vdots & m+1 \end{array} \\ \begin{bmatrix} 1 & a_{2,1} & a_{3,1} & \vdots\vdots\vdots & a_{\mathrm{m+1,1}} \\ 0 & a_{2,2} & a_{3,2} & \vdots\vdots\vdots & a_{\mathrm{m+1,2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & a_{2,\mathrm{n}} & a_{3,\mathrm{n}} & \vdots\vdots\vdots & a_{\mathrm{m+1,n}} \\ 0 & 1 & 0 & \vdots\vdots\vdots & 0 \\ 0 & 0 & 1 & \vdots\vdots\vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots\vdots\vdots & 1 \end{bmatrix} \end{array}$$

Where a_{ij} = - 1or 0 or 1 $2 \leq i \leq (m+1),\ \forall 1 \leq j \leq n$.This matrix is actually the transpose of the coefficient matrix for the Min-cut LP that we see below in equation 3.3.6. The objective function of the Min-cut is also the same as the function in equation 1.3.5. From this, it is easy to see that the dual of the Max-flow problem is Min-cut.

Now we shall formulate the linear program(T) for finding the minimum cut capacity as follows,

$$\text{Minimize} \quad \sum_{(i,j)\in E} c_{ij}\, y_{ij}$$

$$\begin{aligned} &\text{Subject to } -u_1 + u_n \geq 1 \\ &u_i - u_j + y_{ij} \geq 0 \quad \forall (i,j) \in E \\ &u_i \in \ 0,1 \qquad\qquad \forall i \\ &y_{ij} \in \ 0,1 \qquad\qquad \forall i,j \end{aligned} \tag{1.3.6}$$

The solution to the above LPP will result in a cut such that, corresponding to a cut $(S, \overline{S})$ of the network N.

$$\begin{aligned} u_i &= 0 \text{ if vertex } i \in S \\ &= 1 \text{ if vertex } i \in \overline{S} \\ y_{ij} &= 1 \ \ if\ i \in S,\ j \in \overline{S} \\ &= 0 \text{ otherwise} \end{aligned} \tag{1.3.7}$$

Now, relax the integrality constraints on u_i and y_{ij} . The resulting LPP($T^{/}$) will be

$$\text{Minimize} \qquad \sum_{(i,j)\in E} c_{ij}\, y_{ij}$$

$$\text{Subject to } -u_1 + u_n \geq 1 \tag{1.3.8}$$

$$u_i - u_j + y_{ij} \geq 0 \quad \forall (i, j) \in E$$

$$u_i \geq 0 \qquad \forall i$$

$$y_{ij} \geq 0 \qquad \forall i, j$$

In any optimal solution to T. $(u_i, u_j) = (0,0)$ *or* $(1,0)$ *or* $(1,1)$ will imply $y_{ij} = 0$ and $(u_i, u_j) = (0,1)$ imply $y_{ij} = 1$ and hence T will give the capacity of the cut $(S, \overline{S})$.

Now, consider the LPP($T^{/}$). Clealrly, this is the dual of the LPP($R^{/}$). Let B be the coefficient matrix of $T^{/}$ .From Lemma 1.3.1 below, B is unimodular. The column vector b of $T^{/}$ consists either 0's or 1's. Then the optmal solution of $T^{/}$ is also the optimal solution of T.

Now by theorem 6, the optimal values of $R^{/}$ and $T^{/}$ are equal. Hence, the optimal values of R and T are also equal. Therefore, the maximum flow in the network is equal to the minimum cut in the network.

Lemma1.3.1. The coefficient matrix B of the LPP(T) is unimodular.

Proof. Consider the matrix B. Clearly B has m+1 rows and n+m columns. Now, let D and E be two partitions of B, such that D consists of the first n columns and E consists of the second m columns. The matrix D is order $(m+1) X\ n$ and E is of order $(m+1) X\ m$. The matrix D will be as below,

$$\begin{array}{c c} & \begin{array}{cccc} u_1 & u_2 & \vdots\vdots\vdots & u_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ m+1 \end{array} & \begin{bmatrix} 1 & 0 & \vdots\vdots\vdots & -1 \\ a_{2,1} & a_{2,2} & \vdots\vdots\vdots & a_{2,\mathrm{n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{m+1,1}} & a_{\mathrm{m+1,1}} & \vdots\vdots\vdots & a_{\mathrm{m+1,n}} \end{bmatrix} \end{array}$$

Where a_{ij} = - 1or 0 or 1 $2 \leq i \leq (m+1),\ \forall 1 \leq j \leq n$. And the matrix E will be as below.

$$\begin{array}{c c} & \begin{array}{cccc} y_{ij} & \vdots\vdots\vdots & \vdots\vdots\vdots & \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m+1 \end{array} & \begin{bmatrix} 0 & 0 & \vdots\vdots\vdots & 0 \\ 1 & 0 & \vdots\vdots\vdots & 0 \\ 0 & 1 & \vdots\vdots\vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots\vdots\vdots & 1 \end{bmatrix} \end{array}$$

Every row of C contains exactly one 1 and one -1. Every column of D will contain exactly one 1. Now we shall prove this by induction on the size of the sub-matrix. Let U be the square sub-matrix of order k of B. For, k=1, the element can only be either 1 or 0 or -1. Assume that all square sub-matrices of order k-1 have determinant equal to 0 or 1 or -1. Considering for k, the different cases can be as,

(1) U consists of at least one column from D. Every column of D has exactly one 1. Then by deleting that column and the corresponding row, we can get a matrix of order k-1. Hence $|U| = 1$ or 0 or 1.

(2) U consists of columns only form C. Now, if ther exists a row with all zeros or one 1 or one -1, then by induction we can see that the determinant of U will be 1 or 0 or -1. Otherwise, every row of U should contain exactly one 1 and one -1. Then by performing the column addition on all the columns will result in a column with all zeros. Then $|U| = 0$.

2. The Concurrent Multi-Commodity Flow Problem

The Multi-commodity flow problem is a more generalized network flow problem. In a multi-commodity flow problem there are $k \geq 1$ commodities each having its own source and sink pair. Because of the multiplicity of the commodities, the problem to be optimized can be defined in several ways. Hence, there exists several variants of the Multi-commodity flow. The problem we are going to discuss is called Concurrent Multi-commodity Flow Problem (CMFP) [15][12]. In this problem, every i^{th} commodity is assigned a demand D_i. Our objective is to maximize a fraction, such that there exists a flow of D_i units for every commodity i. The previously studied maximum flow problem is a special case of Multi-commodity flow problem in which the number of commodities is one(k = 1). Below, we will give formal definitions and linear program formulation of the problem.

2.1. Preliminaries

Definition2.1.1. (Multi-commodity Network) A Multi-commodity Network is a directed graph G(V,E,c) with vertex set V and an arc set E in which every directed edge $(i, j) \in E$, has a non negative capacity $c(i, j) \geq 0, C : V\ X\ V \to R^+$. There are $k \geq 1$ commodities $K_1, K_{2,\dots,} K_k$. For each commodity i, there is an ordered pair (s_i, t_i) representing the source and sink of that commodity where $(s_i, t_i) \in \mathrm{V\,X}\,V$ and $s_i \neq t_i$.

The flow of a commodity is similar to that of a flow in a single -commodity flow network. Let f^i represent the i^{th} flow of the commodity.

Definition2.1.2. (Flow of a commodity) A flow of a commodity is a mapping $f^i : E \to R$ denoted by $f^i{}_{uv}$ *or* $f^i(u,v)$ subject to the following constraints.

(1) $f^i(u,v) \leq c(u,v)$ for each $(u,v) \in E$ (capacity constraint)

(2) $f^i(u,v) = -f^i(v,u)$ (skew-symmetry)

(3) $\sum_{v \in V} f^i(u,v) = 0 \quad \forall u \in V \setminus \ s_i, t_i$ (conservation of flow)

The value of the flow of a commodity i is given by $|f_i| = \sum_{w \in V} f^i(s_i, w)$.

Definition2.1.3. (Concurrent Multi-commodity Flow Problem) Given a Multi-commodity Network along with demands $D_1, D_2, D_3, ..., D_k$ corresponding to the k commodities, the objective is to assign flow to commoditities so as to maximize a fraction λ such that for every commodity i, the value of the flow of the commodity $|f_i|$ is at least λD_i. The assignment should satisfy the following constraints along with the flow constraints,

$$\sum_{i=1}^{k} f^i(u,v) \leq c(u,v) \quad \forall (u,v) \in E \tag{2.1.1}$$

2.2. Linear Programming Formulation

Below, we give the LP formulation for the Concurrent Multi-commodity Flow problem(CMFP)

Maximize λ

Subject to

$$\begin{aligned} &\sum_{i=1}^{k} f^i(u,v) \leq c(u,v) \quad \forall (u,v) \in E \\ &\sum_{w \in V} f^i(u,w) = 0 \quad \forall 1 \leq i \leq k \quad \forall u \in V - \ s_i, t_i \end{aligned} \tag{2.2.1}$$

$$\sum_{w \in V} f^i(s_i, w) \geq \lambda D_i \quad \forall 1 \leq i \leq k$$

The above formulation is very intuitive and straight forward. We will now see another formulation of the same problem which uses paths. Let P represent the set of all non-trivial paths in the network and P_j be the set of paths corresponding to the commodity j (paths from s_i *to* t_i). Let $x(\alpha)$ be a variable corresponding to every path $\alpha \in P$. Then, he LPP formulation (primal) can be given as,

Maximize λ

Subject to

$$\sum_{\alpha: e \in \alpha} x(\alpha) \leq c(e) \quad \forall e \in E \tag{2.2.2}$$

$$\lambda D_j - \sum_{\alpha \in P_j} x(\alpha) \leq 0 \quad \forall 1 \leq j \leq k$$

$$x(\alpha) \geq 0 \quad \forall \alpha \in P$$

The dual problem for the above linear program can be interpreted as assigning weights (z_j) to the commodities and lengths $(y(e))$ to edges such that for any commodity i the length of every path from s_i *to* t_i should be at least z_i. The length of a path is given as the sum of the lengths of all the edges in that path. The LPP formulation for the dual is as follows,

$$\text{Minimize} \qquad \sum_{e \in E} c(e)\, \mathrm{y}(e)$$

Subject to

$$\sum_{e \in \alpha} \mathrm{y}(e) \geq z_j \quad \forall \alpha \in P_j : \forall j \tag{2.2.3}$$

$$\sum_{1 \leq j \leq k} D_j z_j \geq 1$$

$$l(e) \geq 0 \quad \forall e \in E$$

$$z_j \geq 0 \quad \forall j$$

We shall consider the example in the figure 2.1 with two commodities, say K_1 and K_2. Let the demands be D_1 and D_2 respectively. First, the primal formulation of the problem using paths is given. We shall then derive the dual formulation of the problem.

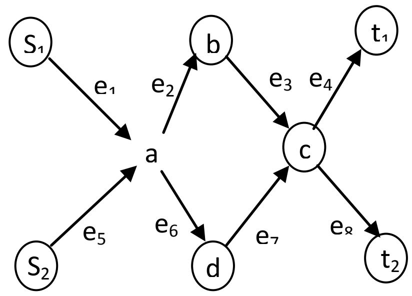

Figure2.1. *An example of a two commodity flow network with unit demands on the commodities.*

In the example figure, there are two paths form source to sink for each of the commodities K_1 and K_2. The edges are labeled $e_1, e_2, ..., e_8$. Let us name the paths with respect to edges as follows,

$$\begin{aligned} &\alpha_1 \rightarrow e_1 - e_2 - e_3 - e_4 \\ &\alpha_2 \rightarrow e_1 - e_6 - e_7 - e_4 \\ &\alpha_3 \rightarrow e_5 - e_2 - e_3 - e_8 \\ &\alpha_1 \rightarrow e_5 - e_6 - e_7 - e_8 \end{aligned} \tag{2.2.4}$$

The paths α_1, α_2 belong to commodity K_1 and the paths α_3, α_4 belong to commodity K_2. The LP formulation for the example is as follows,

$$\begin{gathered}\text{Maximize} \qquad \lambda \\ \text{Subject to}\end{gathered}$$

$$\begin{aligned}
&x(\alpha_1) + x(\alpha_2) \le c(e_1) \\
&x(\alpha_1) + x(\alpha_2) \le c(e_1) \\
&x(\alpha_1) + x(\alpha_4) \le c(e_2) \\
&x(\alpha_1) + x(\alpha_4) \le c(e_3) \\
&x(\alpha_1) + x(\alpha_2) \le c(e_4) \\
&x(\alpha_3) + x(\alpha_4) \le c(e_5) \\
&x(\alpha_3) + x(\alpha_2) \le c(e_6) \\
&x(\alpha_3) + x(\alpha_2) \le c(e_7) \\
&x(\alpha_3) + x(\alpha_4) \le c(e_8) \\
&\lambda D_1 - x(\alpha_1) - x(\alpha_2) \le 0 \\
&\lambda D_2 - x(\alpha_3) - x(\alpha_4) \le 0 \\
&x(\alpha_i) \ge 0 \qquad\qquad \forall 1 \le i \le 4
\end{aligned} \tag{2.2.5}$$

Now, multiply the constraints with the dual variables $(y(e)\, and\ z_j)$ on both sides.

$$\begin{aligned}
&[x(\alpha_1) + x(\alpha_2) \le c(e_1)]y(e_1) \\
&[x(\alpha_1) + x(\alpha_4) \le c(e_2)]y(e_2) \\
&[x(\alpha_1) + x(\alpha_4) \le c(e_3)]y(e_3) \\
&[x(\alpha_1) + x(\alpha_2) \le c(e_4)]y(e_4) \\
&[x(\alpha_3) + x(\alpha_4) \le c(e_5)]y(e_5) \\
&[x(\alpha_3) + x(\alpha_2) \le c(e_6)]y(e_6) \\
&[x(\alpha_3) + x(\alpha_2) \le c(e_7)]y(e_7) \\
&[x(\alpha_3) + x(\alpha_4) \le c(e_8)]y(e_8) \\
&[\lambda D_1 - x(\alpha_1) - x(\alpha_2) \le 0]z_1 \\
&[\lambda D_2 - x(\alpha_3) - x(\alpha_4) \le 0]z_2
\end{aligned} \tag{2.2.6}$$

By combining the equations on the left hand and the right hand sides we get the following inequality,

$$\begin{aligned}
&[x(\alpha_1) + x(\alpha_2)]y(e_1) + [x(\alpha_1) + x(\alpha_4)]y(e_2) + [x(\alpha_1) + x(\alpha_4)]y(e_3) + \\
&[x(\alpha_1) + x(\alpha_2)]y(e_4) + [x(\alpha_3) + x(\alpha_4)]y(e_5) + [x(\alpha_3) + x(\alpha_2)]y(e_6) + \\
&[x(\alpha_3) + x(\alpha_2)]y(e_7) + [x(\alpha_3) + x(\alpha_4)]y(e_8) + [\lambda D_1 - x(\alpha_1) - x(\alpha_2)]z_1 + \\
&[\lambda D_2 - x(\alpha_3) - x(\alpha_4)]z_2 \le \sum_{i=1}^{8} c(e_i) y(e_i)
\end{aligned} \tag{2.2.7}$$

Now, represent the inequality in terms of $x(\alpha)$.

$$\begin{aligned}
&[y(e_1) + y(e_2) + y(e_3) + y(e_4) - z_1]x(\alpha_1) + \\
&[y(e_1) + y(e_6) + y(e_7) + y(e_4) - z_1]x(\alpha_2) + \\
&[y(e_5) + y(e_6) + y(e_7) + y(e_8) - z_2]x(\alpha_3) + \\
&[y(e_5) + y(e_2) + y(e_3) + y(e_8) - z_2]x(\alpha_4) + [D_1 z_1 + D_2 z_2]\lambda \\
&\qquad\qquad\qquad\qquad \le \sum_{i=1}^{8} c(e_i) y(e_i)
\end{aligned} \tag{2.2.8}$$

Now, with respect to the objective function of the primal, the dual can be formulated as below,

$$\text{Minimize} \qquad \sum_{i=1}^{8} c(e_i)\,\mathrm{y}(e_i)$$

Subject to

$$\begin{aligned}
&y(e_1)+y(e_2)+y(e_3)+y(e_4)-z_1 \geq 0\\
&y(e_1)+y(e_6)+y(e_7)+y(e_4)-z_1 \geq 0\\
&y(e_5)+y(e_6)+y(e_7)+y(e_8)-z_2 \geq 0\\
&y(e_5)+y(e_2)+y(e_3)+y(e_8)-z_2 \geq 0\\
&D_1 z_1 + D_2 z_2 \geq 1\\
&y(e_i) \geq 0 \qquad \forall 1 \leq i \leq 8\\
&z_j \geq 0 \qquad \forall 1 \leq j \leq k
\end{aligned} \tag{2.2.9}$$

This is the resulting LPP formulation for the dual of the example we considered in figure 2.1. The dual we have given in equation 2.2.3 is a generalization of this resulting formulation.

The Multi commodity flow problem is very well studied in combinatorics. Unlike single commodity flow, the structural properties of this problem are not well known when the number of commodities is greater than two $(k > 2)$. This problem can be solved in polynomial time using linear programming. However, the problem of finding an integer flow is NP-Complete when $(k \geq 2)$.

3. Conclusions

In this paper we presented the proofs for Konig's theorem and Max-flow Min-cut theorem using a complete different technique based on the total unimodularity property of the coefficient matrix in their linear program formulation. Finally, we have briefly discussed about Multi-commodity flow and the Concurrent Multi-commodity Flow Problem (CMFP). Many more primal-dual relations exist in graph theory and the approach generalize to investigation into these relations and discovering LP based proofs for those Min-Max relations in graph theory. Hence, this approach is a general tool and the results presented here are just sample cases. We have also attempted to apply technique called Lagrangian relaxation [7] from linear programming to some of these relations in order to gain some insights into its effectiveness. A possibility is to apply different techniques in combination and try to investigate the outcome which could lead to interesting observations.

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