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Complementary Acyclic Domination Chain in Graphs

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Abstract: *Let G be a simple graph. A subset S of V(G) is called an independent complementary acyclic set (Independent c-a set) of G if S is independent and <V − S> is acyclic. The existence of independent c-a sets in a graph is not guaranteed. For example, K_n, **n** ≥ 4 and W_n, n even do not have independent c-a sets. But there are graphs which admit c-a independent sets like trees, $K_n(n\geq 4)$ - free graphs , C_n (n ≥ 3), bipartite graphs and W_n , n odd.* ***In this paper, we consider graphs which admit independent c-a sets.*** *Property of being independent and complementary acyclic is neither hereditary nor superhereditary. The minimum cardinality of a maximal independent c-a set of G is called the independent c-a domination number of G and is denoted by $i_{c-a}(G)$. The maximum cardinality of a maximal independent c-a set of G is called the independent c-a number of G and is denoted by $\beta_{c-a}(G)$. A subset S of V(G) is called a c-a dominating set of G if S is dominating and <V-S> is acyclic. It is proved in this paper that a maximal independent c-a set is a minimal c-a dominating set. A study of c-a irredundant set in a graph is initiated and c-a domination chain is found*

Keywords: *Independent c-a domination number, independent c-a set, c-a irredundant set, c-a domination chain.*

Mathematics Subject Classification: *04569*

1. INTRODUCTION

Let G = (V, E) be a finite simple graph. One of the fastest growing areas in graph theory is the theory of domination. A subset S of V is a dominating set of G if every vertex in V-S is adjacent to some vertex in S. A subset S is independent if the subgraph induced by S namely $< S >$ has no edge. A subset S is an irredundant set if S satisfies the condition for minimality of a dominating set. Dominating sets are superhereditary, independent sets are hereditary and irredundant sets are also hereditary. Each parameter is associated with a lower and upper number. Thus we have six positive integers namely γ, Γ; i and β_o ; i_r and IR. In the paper [6], Cockayne et al present several results on the chain of inequalities connecting the six numbers γ, Γ, I, β_o, i_r, and IR. The chain $i_r \leq \gamma \leq i \leq \beta_o \leq \Gamma \leq IR$ is called the domination chain of a graph. In [1], Arumugam and Subramanian extended the domination chain by introducing a new parameter called the independence saturation number denoted by IS. It was proved that IS lies between i and β_o (ie. $i \leq IS \leq \beta_o$). Given an integer sequence (a,b,c,d,e,f) does there exist a graph G such that $ir(G) = a$, $\gamma(G) = b$, $i(G) = c$, $\beta_o(G) = d$, $\Gamma(G) = e$ and $IR(G) = f$? If such a graph exists then (a, b, c, d, e, f) is called a dii-sequence. It has been proved in [3] that a sequence (a,b,c,d,e,f) of positive integers is a dii-sequence if and only if (i) $a \leq b \leq c \leq d \leq e \leq f$.(ii) a=l implies that c = 1 (iii) d = 1 implies that f = 1 (iv) $b \leq 2a - 1$. In [1], finding the necessary and sufficient conditions for a sequence of seven positive integers to be the elements of an extended domination chain is given as a problem.

Since its first publication in 1978, the inequality chain $ir \leq \gamma \leq i \leq \beta_o \leq \Gamma \leq$ IR has been the focus of more than hundred research papers. Many attempts have been made since 2007 to introduce saturation parameters. This study of inequality chain has given inspiration to study similar types in other types of domination. For example, strong domination where vertices outside the set are dominated strongly by vertices inside the set leads to a strong domination chain.

In this paper complementary aylic domination is considered. The problem of finding an inequality chain for complementary aylic domination is studied. Independent sets exist in any graph but complementary acyclic independent sets may not exist in all graphs. So we restrict our attention to the class of graphs which admit independent c-a sets. In this paper necessary condition for the sequence of six positive integers to be a c-a domination sequence is given. The question of finding necessary and sufficient condition for a set of positive integers to form a c-a domination chain is suggested as an open problem.

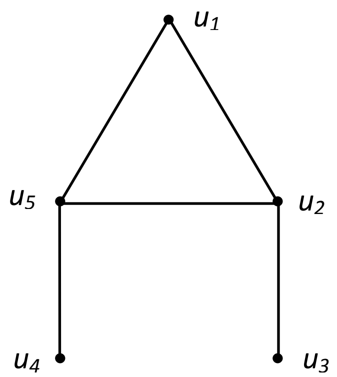
2. Results

Definition2.1. Let G be a simple graph. A subset S of V(G) is called an independent complementary acyclic set (independent c-a set) of G if S is independent and <V-S> is acyclic.

Remark2.2. Property of being independent and complementary acyclic is neither hereditary nor superhereditary.

Example2.3

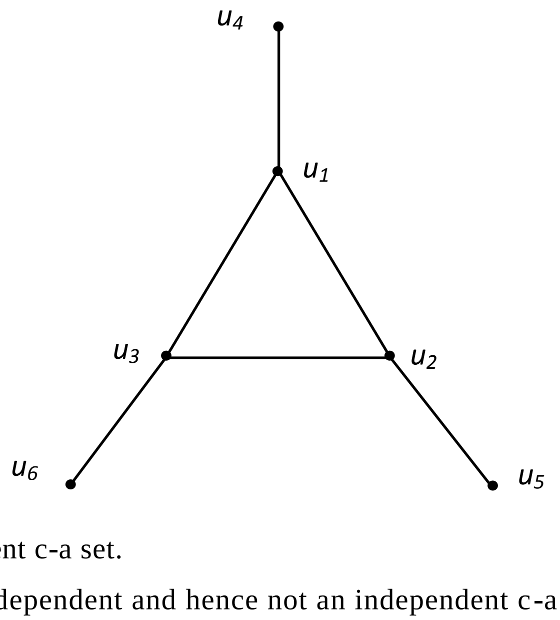
Consider the graph G:



$\{u_1, u_4\}$ is an independent c-a set.

$\{u_4\}$ is a subset which is not complementary acyclic.

Example 2.4



Consider the graph G:

$\{u_3,u_4,u_5\}$ is an independent c-a set.

$\{u_2,u_3,u_4,u_5\}$ is not an independent and hence not an independent c-a set.

Theorem 2.5 *Let G be a simple graph containing $K_n(n \geq 4)$ as an induced subgraph. Then G does not admit independent c-a sets.*

Proof:

Let G be a simple graph containing $K_n(n \geq 4)$ as an induced subgraph. Suppose S is an independent set of G. S can contain atmost one vertex from the induced K_n. Since $n \geq 4$, atleast 3 vertices of the induced K_n belong to V-S and these vertices form a cycle. Therefore V-S is not acyclic. Thus G does not contain any independent c-a set.

Remark 2.6 A graph G admits independent c-a sets if and only if V(G) can be partitioned into two subsets V_1 and V_2 where V_1 is independent and V_2 is acyclic.

Remark 2.7 The Cartesian product $C_5 \square C_4$ of graphs C_5 and C_4 is triangle free and contains independent sets whose complement is acyclic. But there are also triangle free graphs which do not admit independent c-a sets. For example, the Mycielskian of the graph C_5 is triangle free but does not contain independent c-a sets.

Problem: Characterize graphs which admit independent c-a sets.

Theorem 2.8 A maximal independent complementary acyclic set of G is a maximal independent set of G and is also a minimal complementary acyclic dominating set of G.

Proof: Let S be a maximal independent c-a set of G.Then S is a maximal independent set of G. **For**: Let $x \in V - S$. Suppose x is not adjacent with any vertex of S. Then $S \cup \{x\}$ is independent and complementary acyclic, a contradiction. Therefore S is a maximal independent set of G. Also for any $x \in V - S$, x is adjacent with some vertex of S. Therefore S is a c-a dominating set of G. S is a minimal c-a dominating set of G, since for any $u \in S$, $S - \{u\}$ is not a dominating set of G.

Definition 2.9 The minimum cardinality of a maximal independent c-a set of G is called the independent c-a domination number of G and is denoted by $i_{c\text{-}a}(G)$

Definition 2.10 The maximum cardinality of a maximal independent c-a set of G is called the independent c-a number of G and isdenoted by $\beta_{c\text{-}a}(G)$.

We have the chain $\gamma_{c\text{-}a}(G) \leq i_{c\text{-}a}(G) \leq \beta_{c\text{-}a}(G) \leq \Gamma_{c\text{-}a}(G)$.

Remark 2.11 Property of c-a domination is superhereditary.

Theorem 2.12 Let G be a simple graph. Let S be a c-a dominating set of G. Then S is minimal if and only if for every $u \in S$, one of the following holds.

(i) $pn[u; S] \neq \Phi$ where pn [u; S] is private neighbor of u. (ii) $\langle (V\text{-}S) \cup \{u\} \rangle$ contains a cycle.

Definition 2.13 A subset S of V(G) is called a complementary acyclic irredundant set (c-a irredundant set) of G if for every $u \in S$, $pn[u; S] \neq \varphi$ or $\langle V - S \rangle \cup \{u\}$ contains a cycle.

Proposition 2.14 Complementary acyclic irredundance is a hereditary property.

Proof: Let S be a c-a irredundant set of G. Let T be a subset of S. Let $u \in T$. Then $u \in S$.Therefore $pn[u; S] \neq \varphi$ or $\langle V\text{-}S \rangle \{u\}$ contains a cycle. Therefore $pn[u; T] \neq \varphi$ or $(V - T) \cup \{u\}$ *which* contains $(V - S) \cup \{u\}$ contains a *cycle*. Therefore T is a c-a irredundant set of G. Therefore complementary acyclic irredundance is a hereditary property.

Theorem 2.15 Every minimal complementary acyclic dominating set of G is a maximal complementary acyclic irredundant set of G.

Proof: Let S be a minimal c-a dominating set of G. Therefore S is a c-a irredundant set of G. Suppose S is not maximal. Therefore S is not 1 - maximal. Then there exists $u \in V - S$ such that $S \cup \{u\}$ is a c-a irredundant set of G. For any $v \in S \cup \{u\}$, $pn[v; S \cup \{u\}] \neq \varphi$ or $(V - (S \cup \{u\})) \cup \{u\}$ contains a cycle. Take $v = u$.Then $pn[u; S \cup \{u\}] \neq \varphi$ or $(V - (S \cup \{u\})) \cup \{u\}$ contains a cycle. Let $w \in pn[u; S \cup \{u\}]$. That is w is adjacent with u but not adjacent with any vertex of $S \cup \{u\} - \{u\}$. That is w is not adjacent with any vertex of S. Let $w \in S$. Then $w \in N[u] - N[S \cup \{u\} - \{u\}]$. Therefore $w \in N(u)$ and $w \notin N[S]$. Since $w \in S$, $w \in N[S]$, a contradiction. Therefore $w \notin S$. Therefore S does not dominate w, a contradiction, since S is a dominating set of G.Therefore $pn[u; S \cup \{u\}] = \varphi$. Suppose $(V - (S \cup \{u\})) \cup \{u\}$ contains a cycle. That is $\langle V - S \rangle$ contains a cycle, a contradiction, since S is a c-a dominating set. Thus if $S \cup \{u\}$ is a c-a irredundant set of G for some $u \in V - S$, then *pn [u; S∪{u}]* = φ and (V– (S∪{u}))∪{u} contains a cycle, a contradiction. Therefore S is a maximal c-a irredundant set of G.

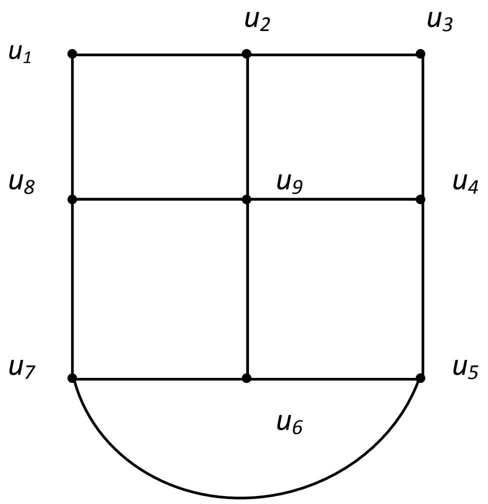
Definition 2.16 The maximum (minimum) cardinality of a c-a irredundant set of G is called upper c-a irredundance (c-a irredundance) number of G and is denoted by $IR_{c\text{-}a}(G)(ir_{c\text{-}a}(G))$.

Corollary 2.17 From the above we get that

$ir_{c\text{-}a}(G) \leq \gamma_{c\text{-}a}(G) \leq i_{c\text{-}a}(G) \leq \beta_{c\text{-}a}(G) \leq \Gamma_{c\text{-}a}(G) \leq IR_{c\text{-}a}(G)$.

Example 2.18

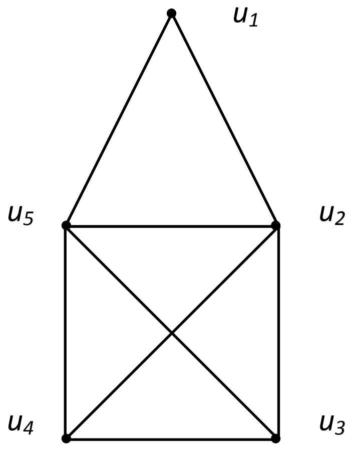
Consider the graph G:



$S = \{u_2, u_5, u_8, u_9\}$ is a c-a irredundant set.

Example 2.19

Consider graph G:

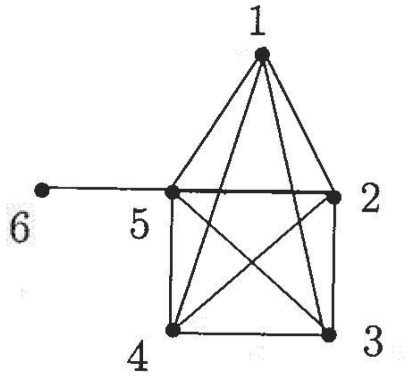


$\{u_3, u_5\}$ is the c-a dominating set. $\{u_3, u_5\}$ is an irredundant c-a set in which u_3 has no private neighbor.

Observation 2.20 In a complete graph of order greater than or equal to four, every c-a dominating set is a c-a irredundant set and every vertex in a c-a dominating set has no private neighbor.

Example 2.21

G:

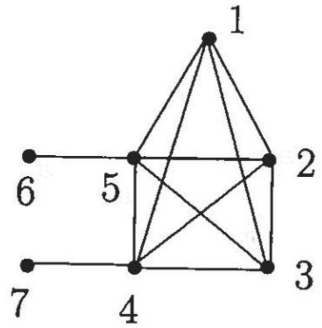


{1, 2, 3} is a maximal c-a irredundant set of G of minimum cardinality.

{1, 2, 3} is not a c-a dominating set, since 6 is not dominated by {1, 2, 3}.

Example 2.22

G:



{1,2,3} is ir_{c-a}- set of G.

{1,4, 5} is a γ_{c-a}-set of G.

In the following, a result similar to the one established in domination is proved.

Theorem 2.23 Let G be a simple graph.

Then $\gamma_{c-a}(G)/2 < ir_{c-a}(G) \leq \gamma_{c-a}(G) \leq 2ir_{c-a}(G) - 1$

Proof: Let G be simple graph .Let $ir_{c-a}(G) = k$. Let $S = \{u_1, u_2, \ldots, u_k\}$ be an ir_{c-a}-set of G.

Let $S_1 = \{v_1, v_2, \ldots, v_k\}$, where $v_i = u_i$ if u_i forms a cycle with V-S or v_i is a private neighbour of u_i. Let $T = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$. Then $|T| \leq 2k$. Suppose $w \in V - T$ is not dominated by T. Therefore $w \notin N[u_i]$ for every i, $1 \leq i \leq k$. Suppose V–T contains a cycle. Therefore $V - S \supseteq V - T$. Therefore V–S contains a cycle, a contradiction, since S is a ir_{c-a}-set of G. Since w $\notin N[u_i]$ for every i, $1 \leq i \leq k$, $pn[w; S \cup \{u\}] \neq \varphi$.

Consider $pn[u_i, S \cup \{w\}]$, $1 \leq i \leq k$. Then $pn[u_i, S \cup \{w\}] = pn[u_i, S]$. If $pn[u_i, S] \neq \varphi$, then $pn[u_i, S \cup \{w\}] \neq \Phi$. If $pn[u_i; S] = \Phi$, then $(V - S) \cup \{ui\}$ contains a cycle. $(V - S \cup \{w\}) \cup \{u_i\} = (V - S) \cup \{u_i\}$ and hence $(V - S \cup \{w\}) \cup \{u_i\}$ contains a cycle.Therefore $(S \cup \{w\})$ is a c-a irredundant set of G, a contradiction to the assumption that S is a maximal c-a irredundant set of G. Therefore T isa c-a dominating set of G. Therefore $\gamma_{c-a}(G) \leq |T| \leq 2k$. Therefore $\gamma_{c-a}(G)/2 \leq k = ir_{c-a}(G)$. Let $T \neq S$.If T is a minimal c-a dominating set, then T is a maximal ir_{c-a}-set containing S, acontradiction, since S is maximal. Therefore T is not minimal. Therefore $\gamma_{c-a}(G) < |T| \leq 2k$.

Therefore $\gamma_{c-a}(G)/2 < k$. That is $\gamma_{c-a}(G)/2 < ir_{c-a}(G)$. When T=S, then $|T| = |S| = ir_{c-a}(G)$. But T is a minimal c-a dominating set of G. $\gamma_{c-a}(G) < ir_{c-a}(G)$. But $ir_{c-a}(G) < \gamma_{c-a}(G)$. Therefore $\gamma_{c-a}(G) = ir_{c-a}(G)$. Therefore $\gamma_{c-a}(G)/2 = ir_{c-a}(G)/2 < ir_{c-a}(G)$. T is a c-a dominating set of G but not a minimal c-a dominating set of G.Therefore $\gamma_{c-a}(G) < |T| \leq 2k$. Therefore $\gamma_{c-a}(G) \leq 2k - 1 = 2ir_{c-a}(G)) - 1$.

Theorem 2.24 If *a sequence a, b, c, d, e, f of positive integers is a c-a domination sequence, then*

(i) $a \leq b \leq c \leq d \leq e \leq f$.

(ii) $b < 2a - 1$

(iii) If $a = 1$, *then* $c = 1$.

(iv) If $d = 1$, *then* $f = 1$.

Proof: Suppose a, b, c, d, e, f is a c-a domination sequence. Then (i) and (ii) are easy consequences. Suppose a = 1.Then there exists an c-a irredundant set, say S of cardinality 1. Let $S = \{u\}$. If $v \in V - S$ is not dominated by u, then $S_1 = \{u, v\}$ is an independent set and hence a c-a irredundant set of G containing S, a contradiction, since S is a maximal c-a irredundant set of G. Therefore u is adjacent with every vertex of V–S. Therefore S is a dominating set of G. Suppose V-S contains a cycle. Let w be a vertex of that cycle. Since that cycle is of length greater than or equal to 3, there exist vertices x, y which lie on the cycle. Consider T = {u, w}. Both (V– T)∪{u} and (V –T)∪{w} contain cycles. Therefore T is a c-a irredundant set of G containing S, a contradiction. Therefore S = {u} is a c-a dominating set of G.

Therefore $ir_{c-a}\{G) = \gamma_{c-a}(G) = 1$.Since S is an independent set which is maximal, S is a maximal c-a independent set of G. Since $\backslash S\backslash = 1$, S is a c-a maximal independent set ofG of minimum cardinality.Therefore $i_{c-a}(G) = 1$.Therefore $ir_{c-a}(G) = i_{c-a}(G) = 1$.Therefore if a = 1, then c = 1.

Suppose $\beta_{c-a}(G) = 1$. Let $S = \{u\}$ be a β_{c-a}-set of G. Then u is a full degree vertex and $V – \{u\}$ is acyclic. Since $\beta_{c-a}(G) = 1$, $ir_{c-a}(G) = 1$ and $\gamma_{c-a}(G) = 1$.Suppose $IR_{c-a} \geq 2$. Let $S_1 = \{v_1,v_2, ..., v_k\}$ be an IR_{c-a}-set of G. Let S_2 be a maximum independent subset of S_1.Then S_2 is complementary acyclic and S_2 is independent.Therefore $|S_2| \leq \beta_{c-a}(G) = 1$.Therefore $|S_2|= 1$. Then S_2 is a β_{c-a}-set of G and S_2 is a maximal independent set of G.Therefore $S_2 = \{u\}$ is a full degree vertex of G. Also $u \in S_1$. Forany $v \neq u$ in S_1, v is adjacent with u and <V-{u}> is acyclic. Therefore $pn[v; S_1] =\varphi$ and $<(V - S_1)\cup\{u\}>$ is acyclic. Therefore S_1 cannot contain any vertex other than u. Therefore $\Gamma_{c-a}(G) = 1 = \beta_{c-a}(G)$.Therefore, if d = l, then f=1.

Problem: Find necessary and sufficient condition for a set of positive integers a, b, c, d, e, and f with $a \leq b \leq c \leq d \leq e \leq f$ *to* form a complementary acyclic domination chain.

3. Conclusion

It is a very important concept in Graph theory. It has varied applications. Also, domination has many variations. One of them is complementary acyclic domination. The minimum/maximum parameters form a chain called the domination chain which is of high interest in domination theory. The present investigation extends this inequality to the new type of domination namely complementary acyclic domination in the class of graphs which admit independent c-a sets. Each type of c-a domination has its applications in social net work theory. For example, independent c-a domination implies the existence of a good governing council where the members do not have unnecessary interactions and the people in complementary do not have cyclic interactions. Indication for further research is also given.

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