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The Gaussian Fibonacci Skew-Circulant Type Matrices

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Abstract: *Let $A_{r,n}$ be a Gaussian Fibonacci skew-circulant matrix, and $A'_{r,n}$ be a Gaussian Fibonacci left skew-circulant matrix, and both of the first rows are $(G_{r+1},\ G_{r+2},...,G_{r+n})$, where G_{r+n} is the $(r+n)$th Gaussian Fibonacci number, and r is a nonnegative integer. In this paper, by constructing the transformation matrices, the explicit determinants of A and A' are expressed. Moreover, we discuss the singularities of these matrices and the inverse matrices of them are obtained.*

Keywords: *Gaussian Fibonacci, skew-circulant matrix, determinant, singularity, inverse*

1. INTRODUCTION

Circulant matrices have a wide range of applications, for examples in coding theory, image processing, self-regress design and so on. Recently, some authors gave the explicit determinant and inverse of the circulant and skew-circulant involving famous numbers. Jiang and Yao gave determinants, norm, and spread of skew circulant type matrices involving any continuous Lucas numbers in [1]. In [2], Jiang and Hong presented exact determinants of some special circulant matrices involving four kinds of famous numbers. An explicit form of the inverse of a particular circulant matrix is presented by Cambini in [3]. Jiang and Li [4] discussed the nonsingularity of the circulant type matrix and gave the explicit determinant and inverse matrices. In [5], the nonsingularity of the skew circulant type matrices is studied and the explicit determinants and inverse matrices of these special matrices are also presented. Besides, authors gave four kinds of norms and bounds for the spread of these matrices separately. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses in [6]. Jiang et al. [7] considered circulant type matrices with the k-Fibonacci and k-Lucas numbers and presented the explicit determinant and inverse matrix by constructing the transformation matrices.

The Gaussian Fibonacci sequence [8, 9] is defined by the following recurrence relations:

$$G_{n+1}=G_n+G_{n-1}, n\geq 1,$$

with the initial condition $G_0=\mathrm{i}$, $G_1=1$. The G_n is given by the formula

$$G_n=\frac{(1-\mathrm{i}\beta)\alpha^n+(\mathrm{i}\alpha-1)\beta^n}{\alpha-\beta}=\frac{\alpha^n-\beta^n+(\alpha^{n-1}-\beta^{n-1})\mathrm{i}}{\alpha-\beta},$$

where α and β are the roots of the characteristic equation $x^2-x-1=0$.

In this paper, the skew-circulant type matrices, including the skew-circulant, left skew-circulant. We give the explicit determinants of them and inverse matrices under the condition of invertible.

Definition 1. A Gaussian Fibonacci skew-circulant matrix, denoted by $\mathrm{SCirc}(G_{r+1},...,G_{r+n})$, is a matrix of the form:

$$\begin{bmatrix} G_{r+1} & G_{r+2} & \cdots & G_{r+n-1} & G_{r+n} \\ -G_{r+n} & G_{r+1} & G_{r+2} & \cdots & G_{r+n-1} \\ \vdots & -G_{r+n} & G_{r+1} & \ddots & \vdots \\ -G_{r+3} & \vdots & \ddots & \ddots & G_{r+2} \\ -G_{r+2} & -G_{r+3} & \cdots & -G_{r+n} & G_{r+1} \end{bmatrix}_{n\times n},$$

where r is a nonnegative integer.

Definition 2. A Gaussian Fibonacci left skew-circulant matrix, denoted by $\mathrm{SLCirc}(G_{r+1}, ..., G_{r+n})$, is a matrix of the form

$$\begin{bmatrix} G_{r+1} & G_{r+2} & G_{r+3} & \cdots & G_{r+n} \\ G_{r+2} & G_{r+3} & \cdots & G_{r+n} & -G_{r+1} \\ G_{r+3} & & & & \vdots \\ \vdots & G_{r+n} & -G_{r+1} & \cdots & -G_{r+n-2} \\ G_{r+n} & -G_{r+1} & \cdots & -G_{r+n-2} & -G_{r+n-1} \end{bmatrix}_{n\times n}.$$

Lemma 3. [10] If $M = \mathrm{Circ}_\omega(a_1, a_2, \ldots, a_n)$, then

$$\lambda_k = \textstyle\sum_{j=1}^{n} a_j \xi_k^{j-1},$$

and

$$\det M = \textstyle\prod_{k=1}^{n} \sum_{j=1}^{n} a_j \xi_k^{j-1}$$

where ξ_k $(k = 0, 1, ..., n-1)$ are the roots of the equation $x^n - \omega = 0$.

Lemma 4. [10] Let $M = \mathrm{Circ}_\omega(a_1, \ldots, a_n)$. Then M is nonsingular if and only if $(f(x), g(x)) = 1$, where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$ and $g(x) = x^n - \omega$ for $\omega \neq 0$.

2. Determinant and Inverse of Gaussian Fibonacci Skew-Circulant Matrices

In this section, let $A_{r,n} = \mathrm{SCirc}(G_{r+1}, ..., G_{r+n})$ be a skew-circulant matrix. We first give the explicit determinant of the matrix $A_{r,n}$, then discuss the singularity of it, and according to the case, present the inverse of $A_{r,n}$.

Theorem 5. Let $A_{r,n} = \mathrm{SCirc}(G_{r+1}, ..., G_{r+n})$ be a skew-circulant matrix. Then we have

$$\begin{aligned}&\det A_{r,n}\\ &= G_{r+1} \cdot \left[(G_{r+1} + \tfrac{G_{r+2}}{G_{r+1}} G_{r+n}) + \textstyle\sum_{k=1}^{n-2} (\tfrac{G_{r+2}}{G_{r+1}} G_{r+k+1} - G_{r+k+2}) (\tfrac{-G_{r+n} - G_r}{G_{r+1} + G_{r+n+1}})^{n-(k+1)} \right]\\ &\times (G_{r+1} + G_{r+n+1})^{n-2}.\end{aligned}$$

Furthermore, A is singular if and only if $G_{r+1} + G_{r+n+1} + (G_r + G_{r+n})\varepsilon_k = 0$ and $(1 - \alpha\varepsilon_k)(1 - \beta\varepsilon_k) \neq 0$, where G_{r+n} is the $(r+n)$ th Gaussian Fibonacci number , $\varepsilon_k = \cos\frac{(2k+1)\pi}{n} + \mathrm{i}\sin\frac{(2k+1)\pi}{n}$.

Proof. We give the explicit determinant of the matrix $A_{r,n}$ firstly. Let

$$\Lambda = \begin{pmatrix} 1 & & & & & & \\ \frac{G_{r+2}}{G_{r+1}} & & & & & & 1 \\ 1 & & & & & 1 & -1 \\ 0 & & & 0 & 1 & -1 & -1 \\ \vdots & & & & & & \\ 0 & & & 1 & & & \\ 0 & & 1 & -1 & & 0 & \\ 0 & 1 & -1 & -1 & & & \end{pmatrix}, \Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & (\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices, then we have

$$\Lambda A_{r,n}\Pi_1 = \begin{pmatrix} G_{r+1} & f'_{r,n} & G_{r+n-1} & \cdots & G_{r+3} & G_{r+2} \\ 0 & f_{r,n} & a_n & \cdots & a_4 & a_3 \\ 0 & 0 & b & & 0 & 0 \\ 0 & 0 & c & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & & b & 0 \\ 0 & 0 & 0 & & c & b \end{pmatrix},$$

where

$f'_{r,n} = \sum_{k=1}^{n-1} G_{r+k+1}(\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}$,

$f_{r,n} = (G_{r+1} + \frac{G_{r+2}}{G_{r+1}}G_{r+n}) + \sum_{k=1}^{n-2}(\frac{G_{r+2}}{G_{r+1}}G_{r+k+1} - G_{r+k+2})(\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}$,

$a_n = \frac{G_{r+2}}{G_{r+1}}G_{r+n-1} - G_{r+n}$,

$a_4 = \frac{G_{r+2}}{G_{r+1}}G_{r+3} - G_{r+4}$,

$a_3 = \frac{G_{r+2}}{G_{r+1}}G_{r+2} - G_{r+3}$,

$b = G_{r+1} + G_{r+n+1}, c = G_r + G_{r+n}$.

So we can get

$$\begin{aligned} &\det\Lambda \det A_{r,n} \det \Pi_1 \\ &= G_{r+1}f_{r,n}(G_{r+1}+G_{r+n+1})^{n-2} \\ &= G_{r+1}\cdot\left[(G_{r+1} + \frac{G_{r+2}}{G_{r+1}}G_{r+n}) + \sum_{k=1}^{n-2}(\frac{G_{r+2}}{G_{r+1}}G_{r+k+1} - G_{r+k+2})(\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}\right] \\ &\times (G_{r+1}+G_{r+n+1})^{n-2}, \end{aligned}$$

while $\det\Lambda = (-1)^{\frac{(n-1)(n-2)}{2}}$, $\det\Pi_1 = (-1)^{\frac{(n-1)(n-2)}{2}}$, hence, we have

$$\begin{aligned} &\det A_{r,n} \\ &= G_{r+1}\cdot\left[(G_{r+1} + \frac{G_{r+2}}{G_{r+1}}G_{r+n}) + \sum_{k=1}^{n-2}(\frac{G_{r+2}}{G_{r+1}}G_{r+k+1} - G_{r+k+2})(\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}\right] \\ &\times (G_{r+1}+G_{r+n+1})^{n-2}. \end{aligned}$$

Next, we discuss the singularity of the matrix A.

The roots of polynomial $g(x) = x^n + 1$ are $\eta\varepsilon_k (k = 0, 1, 2, ..., n-1)$, where $\eta = |-1|^{\frac{1}{n}} = 1$, $\varepsilon_k = \cos\frac{(2k+1)\pi}{n} + \mathrm{i}\sin\frac{(2k+1)\pi}{n}$. By Lemma 3, the eigenvalues of A are

$$\begin{aligned} f(\varepsilon_k) &= \sum_{j=1}^{n} G_{r+j}(\varepsilon_k)^{j-1} \\ &= \frac{1}{\alpha-\beta}\sum_{j=1}^{n}[(1-\mathrm{i}\beta)\alpha^{r+j} + (\mathrm{i}\alpha-1)\beta^{r+j}](\varepsilon_k)^{j-1} \\ &= \frac{1}{\alpha-\beta}\left[\frac{(1-\mathrm{i}\beta)(1+\alpha^n)\alpha^{r+1}}{1-\alpha\varepsilon_k} + \frac{(\mathrm{i}\alpha-1)(1+\beta^n)\beta^{r+1}}{1-\beta\varepsilon_k}\right] \\ &= \frac{1}{\alpha-\beta}\left[\frac{\alpha^{r+1}-\beta^{r+1}+(\alpha^r-\beta^r)\mathrm{i}}{(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k)} + \frac{\alpha^{r+n+1}-\beta^{r+n+1}+(\alpha^{r+n}-\beta^{r+n})\mathrm{i}}{(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k)} + \frac{\alpha^r-\beta^r+(\alpha^{r-1}-\beta^{r-1})\mathrm{i}}{(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k)}\varepsilon_k + \frac{\alpha^{r+n}-\beta^{r+n}+(\alpha^{r+n-1}-\beta^{r+n-1})\mathrm{i}}{(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k)}\varepsilon_k\right] \\ &= \frac{G_{r+1}+G_{r+n+1}+(G_r+G_{r+n})\varepsilon_k}{(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k)} \end{aligned}$$

$$(k = 0, 1, 2, ..., n-1).$$

By Lemma 4, the matrix A is nonsingular if and only if $f(\varepsilon_k) \neq 0$. That is when $(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k) \neq 0$, A is nonsingular if and only if $G_{r+1} + G_{r+n+1} + (G_r + G_{r+n})\varepsilon_k \neq 0$. When $(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k) = 0$, we have $\varepsilon_k = \frac{1}{\alpha}$ or $\varepsilon_k = \frac{1}{\beta}$. If $\varepsilon_k = \frac{1}{\alpha}$, the eigenvalue of A is

$$\lambda_k = G_{r+1} + G_{r+2}(\varepsilon_k) + \cdots + G_{r+n}(\varepsilon_k)^{n-1}$$
$$= \frac{1}{\alpha-\beta}\left[n\alpha^{r+1} + \frac{\alpha^n-\beta^n}{\alpha^{n-1}}\frac{\beta^{r+1}}{\beta-\alpha}\right] + \frac{1}{\alpha-\beta}\left[n\alpha^{r} + \frac{\alpha^n-\beta^n}{\alpha^{n-1}}\frac{\beta^{r}}{\beta-\alpha}\right]\mathrm{i},$$

for $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $n \in N_+$, $k = 0, 1, ..., n-1$. Then $n\alpha^{r+1} > 0$, $n\alpha^r > 0$, and $\frac{\alpha^n-\beta^n}{\alpha^{n-1}} > 0$.

If r is even number, $\frac{\beta^{r+1}}{\beta-\alpha} > 0$, $Re\lambda_k \neq 0$. If r is odd number, $\frac{\beta^r}{\beta-\alpha} > 0$, $Im\lambda_k \neq 0$. So $\lambda_k \neq 0$. The arguments for $\varepsilon_k = \frac{1}{\beta}$ are similar. Hence, A is nonsingular for $(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k) = 0$.

Thus, the proof is completed.

Lemma 6. Let the matrix $B = [b_{i,j}]_{i,j=1}^{n-2}$ be of the form

$$b_{i,j} = \begin{cases} -G_{r+1} - G_{r+n+1}, & i = j, \\ -G_r - G_{r+n}, & i = j+1, \\ 0, & otherwise, \end{cases}$$

then the inverse $B^{-1} = [b'_{i,j}]_{i,j=1}^{n-2}$ of the matrix B is equal to

$$b'_{i,j} = \begin{cases} \frac{-(-G_{r+n}-G_r)^{i-j}}{(G_{r+1}+G_{r+n+1})^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases}$$

Proof. Let $c_{i,j} = \sum_{k=1}^{n-2} b_{i,k}b'_{k,j}$. Obviously, $c_{i,j} = 0$ for $i < j$. In the case $i = j$, we obtain

$$c_{i,i} = b_{i,i}b'_{i,i}$$
$$= (-G_{r+1} - G_{r+n+1}) \cdot \frac{-1}{G_{r+1}+G_{r+n+1}}$$
$$= 1.$$

For $i \geq j+1$, we have

$$c_{i,j} = \sum_{k=1}^{n-2} b_{i,k}b'_{k,j}$$
$$= b_{i,i-1}b'_{i-1,j} + b_{i,i}b'_{i,j}$$
$$= (-G_r - G_{r+n}) \cdot \frac{-(-G_{r+n}-G_r)^{i-j-1}}{(G_{r+1}+G_{r+n+1})^{i-j}} + (-G_{r+1} - G_{r+n+1}) \cdot \frac{-(-G_{r+n}-G_r)^{i-j}}{(G_{r+1}+G_{r+n+1})^{i-j+1}}$$
$$= 0.$$

Hence, we verify $BB^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $B^{-1}B = I_{n-2}$. Thus, the proof is completed.

Theorem 7. Let $A_{r,n} = \mathrm{SCirc}(G_{r+1}, ..., G_{r+n})$ $(n > 2)$ be a skew-circulant matrix. If A is nonsingular, we have

$$A_{r,n}^{-1}$$
$$= \frac{1}{f_{r,n}}\mathrm{SCirc}\Bigg(1 - \sum_{i=1}^{n-2}\frac{(G_{r+n+2-i}-hG_{r+n+1-i})(-G_{r+n}-G_r)^{i-1}}{(G_{r+1}+G_{r+n+1})^i},$$
$$-h - \sum_{i=1}^{n-2}\frac{(G_{r+n+1-i}-hG_{r+n-i})(-G_{r+n}-G_r)^{i-1}}{(G_{r+1}+G_{r+n+1})^i}, -\frac{G_{r+3}-hG_{r+2}}{G_{r+1}+G_{r+n+1}}, -\frac{(G_{r+3}-hG_{r+2})(-G_{r+n}-G_r)}{(G_{r+1}+G_{r+n+1})^2},$$
$$-\frac{(G_{r+3}-hG_{r+2})(-G_{r+n}-G_r)^2}{(G_{r+1}+G_{r+n+1})^3}, \cdots, -\frac{(G_{r+3}-hG_{r+2})(-G_{r+n}-G_r)^{n-3}}{(G_{r+1}+G_{r+n+1})^{n-2}}\Bigg),$$

where

$h = \frac{G_{r+2}}{G_{r+1}}$,

$f_{r,n} = (G_{r+1} + \frac{G_{r+2}}{G_{r+1}} G_{r+n}) + \sum_{k=1}^{n-2} (\frac{G_{r+2}}{G_{r+1}} G_{r+k+1} - G_{r+k+2})(\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}$.

Proof. According to Theorem 5, when $G_{r+1} + G_{r+n+1} + (G_r + G_{r+n})\varepsilon_k \neq 0$,

A is invertible. Let

$$\Pi_2 = \begin{pmatrix} 1 & -\frac{f'_{r,n}}{G_{r+1}} & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix},$$

where

$x_i = \frac{f'_{r,n}}{f_{r,n}} \frac{G_{r+n+3-i} - \frac{G_{r+2}}{G_{r+1}} G_{r+n+2-i}}{G_{r+1}} + \frac{G_{r+n+2-i}}{G_{r+1}} (i = 3, 4, ..., n)$,

$y_i = -\frac{G_{r+n+3-i} - \frac{G_{r+2}}{G_{r+1}} G_{r+n+2-i}}{f_{r,n}} \ (i = 3, 4, .., n)$,

$f'_{r,n} = \sum_{k=1}^{n-1} G_{r+k+1} (\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}$.

Then we have $\Lambda A_{r,n} \Pi_1 \Pi_2 = \Omega \oplus B$. Where $\Omega = \mathrm{diag}(G_{r+1}, f_{r,n})$ is a diagonal matrix, and $\Omega \oplus B$ is the direct sum of Ω and B. If we denote $\Pi = \Pi_1 \Pi_2$, then we obtain $A_{r,n}^{-1} = \Pi(\Omega^{-1} \oplus B^{-1})\Lambda$.

Since the last row elements of the matrix Π are $0, 1, y_3, y_4, ..., y_{n-1}, y_n$. Hence, by Lemma 6, if let $A_{r,n}^{-1} = \mathrm{SCirc}(u_1, u_2, ..., u_n)$, then its last row elements are given by the following equations:

$u_2 = -\frac{1}{f_{r,n}} \frac{G_{r+2}}{G_{r+1}} - \frac{1}{f_{r,n}} C_n^{(n-2)}$,

$u_3 = -\frac{1}{f_{r,n}} C_n^{(1)}$,

$u_4 = -\frac{1}{f_{r,n}} C_n^{(2)} + \frac{1}{f_{r,n}} C_n^{(1)}$,

$u_5 = -\frac{1}{f_{r,n}} C_n^{(3)} + \frac{1}{f_{r,n}} C_n^{(2)} + \frac{1}{f_{r,n}} C_n^{(1)}$,

$\vdots$

$u_n = -\frac{1}{f_{r,n}} C_n^{(n-2)} + \frac{1}{f_{r,n}} C_n^{(n-3)} + \frac{1}{f_{r,n}} C_n^{(n-4)}$,

$u_1 = \frac{1}{f_{r,n}} - \frac{1}{f_{r,n}} [C_n^{(n-2)} + C_n^{(n-3)}]$.

Let

$$C_n^{(j)} = \sum_{i=1}^{j} \frac{(G_{r+3+j-i} - \frac{G_{r+2}}{G_{r+1}} G_{r+2+j-i})(-G_{r+n}-G_r)^{i-1}}{(G_{r+1}+G_{r+n+1})^i}$$

$$= \sum_{i=1}^{j} \frac{\delta_{j,r}}{(\mu_{2,r})^i} (\mu_{1,r})^{i-1} (j = 1, 2, ..., n-2),$$

then we have

$C_n^{(2)} - C_n^{(1)}$

$= \sum_{i=1}^{2} \frac{\delta_{2,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} - \frac{\delta_{1,r}}{\mu_{2,r}}$

$= \frac{G_{r+3}-\frac{G_{r+2}}{G_{r+1}}G_{r+2}}{(G_{r+1}+G_{r+n+1})^2}(-G_{r+n}-G_r)$

$= \frac{\delta_{1,r}}{(\mu_{2,r})^2}\mu_{1,r},$

$C_n^{(n-2)} + C_n^{(n-3)}$

$= \sum_{i=1}^{n-2} \frac{\delta_{n-2,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} + \sum_{i=1}^{n-3} \frac{\delta_{n-3,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i}$

$= \sum_{i=1}^{n-3} \frac{(G_{r+n+2-i}-\frac{G_{r+2}}{G_{r+1}}G_{r+n+1-i})(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} + \frac{\delta_{1,r}(\mu_{1,r})^{n-3}}{(\mu_{2,r})^{n-2}}$

$= \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i}-\frac{G_{r+2}}{G_{r+1}}G_{r+n+1-i})(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i},$

$C_n^{(j+2)} - C_n^{(j+1)} - C_n^{(j)}$

$= \sum_{i=1}^{j+2} \frac{\delta_{j+2,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} - \sum_{i=1}^{j+1} \frac{\delta_{j+1,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} - \sum_{i=1}^{j} \frac{\delta_{j,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i}$

$= \frac{(G_{r+4}-\frac{G_{r+2}}{G_{r+1}}G_{r+3})(\mu_{1,r})^j}{(\mu_{2,r})^{j+1}} + \frac{G_{r+3}-\frac{G_{r+2}}{G_{r+1}}G_{r+2})(\mu_{1,r})^{j+1}}{(\mu_{2,r})^{j+2}} - \frac{(G_{r+3}-\frac{G_{r+2}}{G_{r+1}}G_{r+2})(\mu_{1,r})^j}{(\mu_{2,r})^{j+1}}$

$= \frac{(G_{r+3}-\frac{G_{r+2}}{G_{r+1}}G_{r+2})(\mu_{1,r})^{j+1}}{(\mu_{2,r})^{j+2}}$

$(j = 1, 2, ..., n-4).$

Hence, if A is nonsingular, we can get

$$A_{r,n}^{-1}$$

$$= \mathrm{SCirc}\left(\frac{1-[C_n^{(n-2)}+C_n^{(n-3)}]}{f_{r,n}}, \frac{-C_n^{(n-2)}-\frac{G_{r+2}}{G_{r+1}}}{f_{r,n}}, -\frac{C_n^{(1)}}{f_{r,n}}, -\frac{C_n^{(2)}-C_n^{(1)}}{f_{r,n}}, -\frac{C_n^{(3)}-C_n^{(2)}-C_n^{(1)}}{f_{r,n}},\right.$$
$$\left.\ldots, -\frac{C_n^{(n-2)}-C_n^{(n-3)}-C_n^{(n-4)}}{f_{r,n}}\right)$$

$$= \frac{1}{f_{r,n}}\mathrm{SCirc}\left(1 - \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i}-\frac{G_{r+2}}{G_{r+1}}G_{r+n+1-i})(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i}, -h - \sum_{i=1}^{n-2} \frac{\delta_{n-2,r}(\mu_{1,r})^{i-1}}{(\mu_{2,r})^i}, -\frac{\delta_{1,r}}{\mu_{2,r}},\right.$$
$$\left.-\frac{\delta_{1,r}}{(\mu_{2,r})^2}\mu_{1,r}, -\frac{\delta_{1,r}}{(\mu_{2,r})^3}(\mu_{1,r})^2, \ldots, -\frac{\delta_{1,r}}{(\mu_{2,r})^{n-2}}(\mu_{1,r})^{n-3}\right)$$

$$= \frac{1}{f_{r,n}}\mathrm{SCirc}\left(1 - \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i}-hG_{r+n+1-i})(-G_{r+n}-G_r)^{i-1}}{(G_{r+1}+G_{r+n+1})^i},\right.$$
$$-h - \sum_{i=1}^{n-2} \frac{(G_{r+n+1-i}-hG_{r+n-i})(-G_{r+n}-G_r)^{i-1}}{(G_{r+1}+G_{r+n+1})^i}, -\frac{G_{r+3}-hG_{r+2}}{G_{r+1}+G_{r+n+1}}, -\frac{(G_{r+3}-hG_{r+2})(-G_{r+n}-G_r)}{(G_{r+1}+G_{r+n+1})^2},$$
$$\left.-\frac{(G_{r+3}-hG_{r+2})(-G_{r+n}-G_r)^2}{(G_{r+1}+G_{r+n+1})^3}, \cdots, -\frac{(G_{r+3}-hG_{r+2})(-G_{r+n}-G_r)^{n-3}}{(G_{r+1}+G_{r+n+1})^{n-2}}\right),$$

where

$h = \frac{G_{r+2}}{G_{r+1}},$

$f_{r,n} = (G_{r+1} + \frac{G_{r+2}}{G_{r+1}}G_{r+n}) + \sum_{k=1}^{n-2}(\frac{G_{r+2}}{G_{r+1}}G_{r+k+1} - G_{r+k+2})(\frac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}.$

This completes the proof.

3. Determinant and Inverse of Gaussian Fibonacci Left Skew-Circulant Matrices

In this section, let $A'_{r,n} = \mathrm{SLCirc}(G_{r+1}, G_{r+2}, ..., G_{r+n})$ be a left skew-circulant matrix. By using the obtained conclusions, we give a determinant formula for the matrix $A'_{r,n}$. Considering the singularity of $A'_{r,n}$, its inverse is also presented.

Theorem 8. Let $A'_{r,n} = \mathrm{SLCirc}(G_{r+1}, G_{r+2}, ..., G_{r+n})$ be a left skew-circulant matrix, then

$$\det A'_{r,n}$$

$$= (-1)^{\frac{n(n-1)}{2}} \cdot G_{r+1} \cdot \left[(G_{r+1} + \tfrac{G_{r+2}}{G_{r+1}} G_{r+n}) + \sum_{k=1}^{n-2} (\tfrac{G_{r+2}}{G_{r+1}} G_{r+k+1} - G_{r+k+2})(\tfrac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}})^{n-(k+1)}\right]$$

$$\times (G_{r+1} + G_{r+n+1})^{n-2}.$$

Furthermore, A' is singular if and only if $G_{r+1} + G_{r+n+1} + (G_r + G_{r+n})\varepsilon_k = 0$ and $(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k) \neq 0$, where G_{r+n} is the $(r+n)$ th Gaussian Fibonacci number, $\varepsilon_k = \cos\frac{(2k+1)\pi}{n} + \mathrm{i}\sin\frac{(2k+1)\pi}{n}$.

Proof. We give the explicit determinant of the matrix A' firstly. The matrix A' can be written as

$$A' = \begin{pmatrix} G_{r+1} & G_{r+2} & \cdots & G_{r+n} \\ G_{r+2} & \cdots & G_{r+n} & -G_{r+1} \\ \vdots & & & \vdots \\ G_{r+n} & -G_{r+1} & \cdots & -G_{r+n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ \vdots & & & & \vdots \\ 0 & 0 & -1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} G_{r+1} & \cdots & G_{r+n-1} & G_{r+n} \\ -G_{r+n} & G_{r+1} & \cdots & G_{r+n-1} \\ \vdots & \ddots & \ddots & \vdots \\ -G_{r+2} & \cdots & -G_{r+n} & G_{r+1} \end{pmatrix}$$

$$= \Gamma^{-1} A.$$

Hence, we have $\det A' = \det \Gamma^{-1} \det A$, where A is a skew-circulant matrix and its determinant can be gotten from Theorem 5, $\det \Gamma = (-1)^{\frac{n(n-1)}{2}}$. So

$$\det A' = \det \Gamma^{-1} \det A$$

$$= G_{r+1}\left[(G_{r+1} + \tfrac{G_{r+2}}{G_{r+1}} G_{r+n}) + \sum_{k=1}^{n-2} (\tfrac{G_{r+2}}{G_{r+1}} G_{r+k+1} - G_{r+k+2})\left(\tfrac{-G_{r+n}-G_r}{G_{r+1}+G_{r+n+1}}\right)^{n-(k+1)}\right]$$

$$\times (G_{r+1} + G_{r+n+1})^{n-2} (-1)^{\frac{n(n-1)}{2}}.$$

Next, we discuss the singularity of the matrix A' . A is singular if and only if $G_{r+1} + G_{r+n+1} + (G_r + G_{r+n})\varepsilon_k = 0$ and $(1-\alpha\varepsilon_k)(1-\beta\varepsilon_k) \neq 0$ by Theorem 5. Furthermore, the matrix Γ is nonsingular. Then the result is obtained.

Theorem 9. Let $A'_{r,n} = \mathrm{SLCirc}(G_{r+1}, G_{r+2}, ..., G_{r+n})\,(n > 2)$ be a left skew-circulant matrix. If $A'_{r,n}$ is invertible, then we have

$$A'^{-1}_{r,n}$$

$$= \frac{1}{f''_{r,n}} \mathrm{SLCirc}\Bigg(-1 - \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i} - hG_{r+n+1-i})(G_{r+n}+G_r)^{i-1}}{(-G_{r+1}-G_{r+n+1})^i}, \frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)^{n-3}}{(-G_{r+1}-G_{r+n+1})^{n-2}},$$

$$\ldots, \frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)^2}{(-G_{r+1}-G_{r+n+1})^3}, \frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)}{(-G_{r+1}-G_{r+n+1})^2}, \frac{G_{r+3}-hG_{r+2}}{-G_{r+1}-G_{r+n+1}},$$

$$-h + \sum_{i=1}^{n-2} \frac{(G_{r+n+1-i} - hG_{r+n-i})(G_{r+n}+G_r)^{i-1}}{(-G_{r+1}-G_{r+n+1})^i}\Bigg),$$

where

$h = \frac{G_{r+2}}{G_{r+1}}$,

$f''_{r,n} = (-G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n}) + \sum_{k=1}^{n-2} (G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1})(\frac{G_{r+n}+G_r}{-G_{r+1}-G_{r+n+1}})^{n-(k+1)}$.

Proof. According to Theorem 8, when $G_{r+1} + G_{r+n+1} + (G_r + G_{r+n})\varepsilon_k \neq 0$, A' is invertible. $A'^{-1} = A^{-1}\Gamma$ and the inverse of A can be gotten from Theorem 7.

So

$$A'^{-1}$$

$$= \frac{1}{f_{r,n}} \mathrm{SLCirc}\Bigg(1 + \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i} - hG_{r+n+1-i})(G_{r+n}+G_r)^{i-1}}{(-G_{r+1}-G_{r+n+1})^i}, -\frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)^{n-3}}{(-G_{r+1}-G_{r+n+1})^{n-2}}, \cdots,$$

$$-\frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)^2}{(-G_{r+1}-G_{r+n+1})^3}, -\frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)}{(-G_{r+1}-G_{r+n+1})^2}, -\frac{G_{r+3}-hG_{r+2}}{-G_{r+1}-G_{r+n+1}},$$

$$h - \sum_{i=1}^{n-2} \frac{(G_{r+n+1-i} - hG_{r+n-i})(G_{r+n}+G_r)^{i-1}}{(-G_{r+1}-G_{r+n+1})^i}\Bigg)$$

$$= \frac{1}{f''_{r,n}} \mathrm{SLCirc}\Bigg(-1 - \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i} - hG_{r+n+1-i})(G_{r+n}+G_r)^{i-1}}{(-G_{r+1}-G_{r+n+1})^i}, \frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)^{n-3}}{(-G_{r+1}-G_{r+n+1})^{n-2}}, \cdots,$$

$$-\frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)^2}{(-G_{r+1}-G_{r+n+1})^3}, \frac{(G_{r+3}-hG_{r+2})(G_{r+n}+G_r)}{(-G_{r+1}-G_{r+n+1})^2}, \frac{G_{r+3}-hG_{r+2}}{-G_{r+1}-G_{r+n+1}},$$

$$-h + \sum_{i=1}^{n-2} \frac{(G_{r+n+1-i} - hG_{r+n-i})(G_{r+n}+G_r)^{i-1}}{(-G_{r+1}-G_{r+n+1})^i}\Bigg),$$

where $h = \frac{G_{r+2}}{G_{r+1}}$,

$f''_{r,n} = (-G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n}) + \sum_{k=1}^{n-2} (G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1})(\frac{G_{r+n}+G_r}{-G_{r+1}-G_{r+n+1}})^{n-(k+1)}$.

The proof is completed.

4. CONCLUSION

We give the explicit determinant of Gaussian Fibonacci skew-circulant matrices by constructing the transformation matrices. According to the relation between skew-circulant matrices and left skew-circulant matrices, the explicit determinant of left skew-circulant matrices is also provided. Considering their singularities, we present the inverse matrices of these matrices.

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