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An Implicit Finite-Difference Method for Solving the Heat-Transfer Equation

Vildan Gülkaç

Department of Mathematics, Faculty of Arts and Science, Kocaeli University,
41380 Umuttepe/ İzmit, Turkey
vgulkac@kocaeli.edu.tr

Abstract: *This article deals with finite- difference schemes of two-dimensional heat transfer equations with moving boundary. The method is suggested by solving sample problem in two-dimensional solidification of square prism. The finite-difference scheme improved for this goal is based on the Douglas equation. The results are devised for a two-dimensional model and crosschecked with results of the earlier authors.*

Keywords: *Heat-transfer equation, Finite-difference, Douglas Equation*

1. Introduction

Heat conduction problems with phase-change occur in many physical applications involving solidification or melting such as making of ice the freezing of food, and the solidification or melting of metals in the casting process. Owing to the nonlinear form of the thermal energy balance at the moving solid-melt interface exact analytical solutions at difficult to obtain except only for a limited number of specific cases. For many practical heat transfer problems it is not possible to obtain a solution by means of analytical techniques. For heat-conduction problems involving complicated boundary conditions, irregular boundaries and variable thermal properties, the exact solutions become impossible or the approximate solutions may not be applicable or may not provide the desired degree of accuracy. The numerical methods of solution are useful for such situations. The finite-difference method is widely used in the solution heat-conduction problems.

Finite difference, finite volume, and finite element methods are some of the wide numerical methods used for PDEs and associated energy equations fort he phase change problems. Finite-difference methods have been used extensively in literature either for simple or simplified geometries.

The literatures with numerous references are given in Crank's [1] book, Landau [2], Beaubouff [3] and Ferris and Hill [4].

A diversity of numerical methods have been developed to solve the moving boundary problems. Öziş and Gülkaç [5] the change of variable method introduced by Boadway presented for solving a two-dimensional moving boundary problem involving convective boundary conditions. Gülkaç [6] two different finite-differences schemes presented for numerical solution of two-dimensional moving boundary problem. Gülkaç and Öziş [7] suggested on a LOD method for the solution of multi-dimensional moving boundary problem.

I.V. Sing et all [8] presented the transient and steady-state solution of two-dimensional heat transfer through the fins using a meshless element free Galerkin method. R.K. Sahoo and V. Prasad [9] presented, application of composite adaptive grid generation and migration scheme for phase-change materials processes. S. J. Yang and W.S. Fu [10] a numerical simulation performed to study the flow structures and heat transfer characteristics of a heated transversely oscillating rectangular cylinder in a crossflow.

In this article, Douglas equation has been used to obtain fully implicit finite-difference equations for two- dimensional heat- transfer equations, and its accuracy was examined by the Fourier series method for stability analysis.

2. TWO-DIMENSIONAL PROBLEM

An infinitely long square prism is initially filled with a fluid at its fusion temperature of unity say. The temperature on its surface is suddenly dropped to zero and is maintained constant throughout. The process of solidification starts from the surface inwards. We shall be interested in finding the temperature distribution in the solidified region along with the determination of the interface position at any instant of time [6].

Let us assume that the square cross-section of the prism extends between $-1 \le x, y \le 1$ If u(x,y,t) denotes temperature at a point (x,y) a some time t, the governing equation for heat-conduction may be written as,

$$u_t = u_{xx} + u_{yy} \text{ in D} \tag{1}$$

Where D is the domain enclosed by fixed boundary

$$f(x, y) \equiv (x^2 - 1)(y^2 - 1) = 0 \text{ where } u = 0 \tag{2}$$

and interface $\phi(x, y, t) = 0$ where $u = 1$ for $t > 0$ (3)

As the nature of curve $\phi(x, y, t) = 0$ representing the solid-liquid interface at any time, t, is known, it has to be determined as part on the solution. Obviously we have,

$$\phi(x, y, t) = f(x, y) = 0 \text{ at t=0} \tag{4}$$

Due to symmetry about the axes and about the diagonals, we need to consider the triangular region. R defined as [6].

3. DESCRIPTION OF THE METHOD

All derivatives can be expressed exactly in term of infinite series of forward, backward or central-differences. For example,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{(\delta_x)^2}(\delta_x^2 u - \frac{1}{12}\delta_x^4 u + \frac{1}{90}\delta_x^6 u + ...) \tag{5}$$

Where the subscript x denotes differencing in the x-direction and the central- differences are

Defined by $\delta_x u_{i,j} = u_{i+1/2,j} - u_{i-1/2,j}$

and

$$\delta_x^2 u_{i,j} = \delta_x(\delta_x u_{i,j}) = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$$

similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{(\delta_y)^2}(\delta_y^2 u - \frac{1}{12}\delta_y^4 u + \frac{1}{90}\delta_y^6 u + ...) \tag{6}$$

Where the subscript y denotes differencing in the y-direction.

In the approximation methods already considered the right-hand side of (5) and (6) have been truncated after the first term. The accuracy of the approximation method will always be improved but this normally increases the number of unknowns in an implicit method and complicates the boundary procedure. For equations involving second-order derivatives, however it is possible to eliminate the fourth-order central-differences yet leave the number of unknowns unchanged.

If the equation (1) is approximated at the point $(i, j, m+1/2)$ by

$$\frac{1}{k}(u_{i,j}^{m+1)}-u_{i,j}^{m})=\frac{1}{2}\left\{\left(\frac{\partial^2 u}{\partial x^2}\right)_{ij}^{m+1}+\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^{m}\right\}+\frac{1}{2}\left\{\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}^{m+1}+\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}^{m}\right\} \tag{7}$$

$$=\frac{1}{2\delta_x^2}\left\{\delta_x^2-\frac{1}{12}\delta_x^4+\frac{1}{90}\delta_x^6+\ldots\right\}\left(u_{i,j}^{m+1}+u_{i,j}^{m}\right)+\frac{1}{2\delta_y^2}\left\{\delta_y^2-\frac{1}{12}\delta_y^4+\frac{1}{90}\delta_y^6+\ldots\right\}\left(u_{i,j}^{m+1}+u_{i,j}^{m}\right) \tag{8}$$

δ_x^6 and δ_y^6 are neglected , and then the terms involving δ_x^4 and δ_y^4 can be eliminated by $\delta_x^4 u \cong -12\delta_x^2(u''-u)$ and $\delta_y^4 u \cong -12\delta_y^2(u''-u)$ from Taylor series expansion and substituting they into equation (8) leads to,

$$u_{i,j}^{m+1}-\frac{1}{2}r\delta_x^2 u_{i,j}^{m+1}-\frac{1}{2}r\delta_x^2(u''-u)_{i,j}^{m+1}-\frac{1}{2}r\delta_y^2 u_{i,j}^{m+1}-\frac{1}{2}r\delta_y^2(u''-u)_{i,j}^{m+1}$$

$$=u_{i,j}^{m}+\frac{1}{2}r\delta_x^2 u_{i,j}^{m}+\frac{1}{2}r\delta_x^2(u''-u)_{i,j}^{m}-\frac{1}{2}r\delta_y^2 u_{i,j}^{m}+\frac{1}{2}r\delta_y^2(u''-u)_{i,j}^{m} \tag{9}$$

where $\frac{k}{\delta_x^2}=\frac{k}{\delta_y^2}=r$.

And $u''\cong\frac{1}{\delta_x^2}(u_{i+1j}-2u_{i,j}+u_{i-1,j}\ )$, $u''\cong\frac{1}{\delta_y^2}(u_{i,j+1}-2u_{i,j}+u_{i,j-1})$ respectively, substituting they into equation (9) leads to

$$u_{i,j}^{m+1}-\frac{1}{2}r\delta_x^2 u_{i,j}^{m+1}-\frac{1}{2}r\delta_x^2\left\{\frac{1}{\delta_x^2}\left(u_{i+1,j}^{m+1}-2u_{i,}^{m+1}+u_{i-1,j}^{m+1}\right)-u_{i,j}^{m+1}\right\}-\frac{1}{2}r\delta_y^2 u_{i,j}^{m+1}$$

$$-\frac{1}{2}r\delta_y^2\left\{\frac{1}{\delta_y^2}\left(u_{i,j+1}^{m+1}-2u_{i,j}^{m+1}-u_{i,j-1}^{m+1}\right)-u_{i,j}^{m+1}\right\}$$

$$=u_{i,j}^{m}+\frac{1}{2}r\delta_x^2 u_{i,j}^{m}+\frac{1}{2}r\delta_x^2\left\{\frac{1}{\delta_x^2}(u_{i+1,j}^{m}-2u_{i,j}^{m}+u_{i-1,j}^{m})-u_{i,j}^{m}\right\}-\frac{1}{2}r\delta_y^2 u_{i,j}^{m}$$

$$+\frac{1}{2}r\delta_y^2\left\{\frac{1}{\delta_y^2}(u_{i,j+1}^{m}-2u_{i,j}^{m}+u_{i,j-1}^{m})-u_{i,j}^{m}\right\} \tag{10}$$

And $\delta_x^2 u_{i,j}=u_{i+1,j}-2u_{i,j}+u_{i-1,j}$, $\delta_y^2 u_{i,j}=u_{i,j+1}-2u_{i,j}+u_{i,j-1}$ respectively, substituting they into equation (10), this gives that,

$$-\frac{1}{2}ru_{i+1,j}^{m+1}-\frac{1}{2}ru_{i-1,j}^{m+1}+(1+r)u_{i,j}^{m+1}-\frac{1}{2}ru_{i,j+1}^{m+1}+ru_{i,j}^{m+1}-\frac{1}{2}ru_{i,j-1}^{m+1}$$

$$=\frac{1}{2}ru_{i+1,j}^{m}+\frac{1}{2}ru_{i-1,j}^{m}+(1-r)u_{i,j}^{m}-\frac{1}{2}ru_{i,j+1}^{m}+ru_{i,j}^{m}-\frac{1}{2}ru_{i,j-1}^{m} \tag{11}$$

Hence, the differential equation can be approximated by the implicit algebraic equation (11), local truncation error of Douglas equation is O(δ_x^4)+O(δ_y^4)+O(k^2).

4. STABILITY OF EQUATIONS

Analyze the stability of the fully implicit difference equation,

$$\frac{1}{k}(u_{i,j}^{m+1)}-u_{i,j}^{m})=\frac{1}{2}\left\{\left(\frac{\partial^2 u}{\partial x^2}\right)_{ij}^{m+1}+\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^{m}\right\}+\frac{1}{2}\left\{\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}^{m+1}+\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}^{m}\right\} \tag{12}$$

Approximating $u_t=u_{xx}+u_{yy}$ at (ph,qk,rt) substitution of $u_{p,q,r}=e^{i\beta ph}e^{i\beta qk}\xi^r$ where $i=\sqrt{-1}$ into the difference equation shows that, and clearly, we need to change our usual notation $u_{i,j}^m$ to $u(ph,qk,rt)=u_{p,q}^r$, and $\frac{k}{\delta_x^2}=\frac{k}{\delta_y^2}=R$ say.

$$-\frac{1}{2}Ru_{p+1,q}^{r+1}-\frac{1}{2}Ru_{p-1,q}^{r+1}+(1+2R)u_{p,q}^{r+1}-\frac{1}{2}Ru_{p,q+1}^{r+1}-\frac{1}{2}Ru_{p,q-1}^{r+1}$$
$$=\frac{1}{2}Ru_{p+1,q}^{r}+\frac{1}{2}Ru_{p-1,q}^{r}+u_{p,q}^{r}-\frac{1}{2}Ru_{p,q+1}^{r}-\frac{1}{2}Ru_{p-1,q}^{r} \tag{13}$$

and, the equation (13) can be written as

$$-\frac{1}{2}R\,e^{i\beta(p+1)h}e^{i\beta qk}\xi^{r+1}-\frac{1}{2}R\,e^{i\beta(p-1)h}e^{i\beta qk}\xi^{r+1}+(1+2R)e^{i\beta ph}e^{i\beta qk}\xi^{r+1}-\frac{1}{2}R\,e^{i\beta ph}e^{i\beta(q+1)k}\xi^{r+1}$$
$$-\frac{1}{2}R\,e^{i\beta ph}e^{i\beta(q-1)k}\xi^{r+1}$$
$$=\frac{1}{2}R\,e^{i\beta(p+1)h}e^{i\beta qk}\xi^{r}+\frac{1}{2}R\,e^{i\beta(p-1)h}e^{i\beta qk}\xi^{r}+e^{i\beta ph}e^{i\beta qk}\xi^{r}-\frac{1}{2}R\,e^{i\beta ph}e^{i\beta(q+1)k}\xi^{r}-\frac{1}{2}R\,e^{i\beta ph}e^{i\beta(q-1)k}\xi^{r} \tag{14}$$

and equation (14) division by $e^{i\beta ph}e^{i\beta qk}\xi^r$ leads to

$$-\frac{1}{2}R\,e^{i\beta h}\xi-\frac{1}{2}R\,e^{-i\beta h}\xi+(1+2R)\xi-\frac{1}{2}R\,e^{-i\beta k}\xi-\frac{1}{2}\mathrm{Re}^{-i\beta k}\,\xi$$
$$=\frac{1}{2}R\,e^{i\beta h}+\frac{1}{2}R\,e^{-i\beta h}+1-\frac{1}{2}R\,e^{-i\beta k}-\frac{1}{2}R\,e^{-i\beta k} \tag{15}$$

$$\xi=\frac{1+\frac{1}{2}R\left[e^{i\beta h}+e^{-i\beta h}-e^{i\beta k}-e^{-i\beta k}\right]}{(1+2R)-\frac{1}{2}R\{e^{i\beta h}+e^{-i\beta h}+e^{i\beta k}-e^{-i\beta k}\}} \tag{16}$$

and then

$$\xi=\frac{1+R\left\{1-2Sin^2(\beta h/2)-1-2Sin^2(\beta k/2)\right\}}{(1+2R)-R\left\{1-2Sin^2(\beta h/2)+1-2Sin^2(\beta k/2)\right\}} \tag{17}$$

$$\xi=\frac{1-2R\left\{Sin^2(\beta h/2)+Sin^2(\beta k/2)\right\}}{(1+2R)-2R\left\{1-Sin^2(\beta h/2)-Sin^2(\beta k/2)\right\}} \tag{18}$$

Clearly $0<|\xi|\le 1$ for all R and all β.

Therefore the equations are unconditionally stable.

5. Numerical Results and Conclusions

We take the initial positions of various isotherms including the interface from some other sources. In the current case, in order to evaluate the certainty of the results obtained from the current method. We also make a start, like Crank and Gupta [11], Radhey et al.[12] and Gülkaç [6] by taking the initial values of the temperature and interface positions from the one parameter integral method of Poots [13].

Also in order to make a comparative study more accurate, the values of various parameters are taken to be the same $\delta_x = \delta_y = 0.1$ and other with $\delta_x = \delta_y = 0.05$. The value of δ_t has been chosen to be 0.0001 and 0.00004 for the first and the second sets respectively. The problem is solved with values δ_t =0.0001 given Table 1.

Table 1. *Comparison of the x- co-ordinate of solid-liquid interface on the x-axis and diagonal*

Time	On the *x*-axis				On the diagonal			
	Crank and Gupta [11]	Radhey et al. [12]	Gülkaç [6]	Present method *t*=0,0001	Crank and Gupta [11]	Radhey et al. [12]	Gülkaç [6]	Present method *t*=0,0001
0,05	0,8125	0.8125	0.7445	0.7445	0.6483	0.6476	0.7300	0.7300
0,10	0,6979	0.6982	0.4312	0.4310	0.5812	0.5642	0.6190	0.6187
0,15	0,6157	0.6156	0.2495	0.2492	0.5103	0.4935	0.5369	0.5365
0,20	0,5473	0.5463	0.1686	0.1686	0.4428	0.4264	0.4341	0.4337
0,25	0,4565	0.4837	0.1243	0.1240	0.3948	0.3642	0.3971	0.3967
0,30	0,4302	0.4244	0.0965	0.0962	0.3351	0.3130	0.3434	0.3430
0,35	0,3767	0.3663	0.0776	0.0772	0.2831	0.2590	0.2833	0.2828
0,40	0,3337	0.3078	0.0640	0.0636	0.2332	0.2176	0.2380	0.2375
0,45	0,2816	0.2495	0.0537	0.0533	0.1947	0.1764	0.1860	0.1855
0,50	-	0.1894	0.0456	0.0450	-	0.1339	0.0878	0.0872
0,55	-	0.1271	0.0392	0.0387	-	0.0899	0.0798	0.0792
0,60	-	0.0562	0.0340	0.0335	-	0.0398	0.0716	0.0709
0,65	-	-	0.0296	0.0290	-	-	0.0633	0.0627
0,70	-	-	0.0260	0.0254	-	-	0.0548	0.0540
0,75	-	-	0.0229	0.0223	-	-	0.0461	0.0452
0,80	-	-	0.0203	0.0197	-	-	0.0372	0.0365
0,85	-	-	0.0180	0.0174	-	-	0.0282	0.0275
0,90	-	-	0.0161	0.0154	-	-	0.0196	0.0189
0,95	-	-	0.0143	0.0136	-	-	0.0190	0.0182
1,00	-	-	0.0115	0.0105	-	-	0.0096	0.0088

A numerical method for a general multi-dimensional boundary problem has been organized.

The finite-difference scheme has been developed to obtain numerical solutions of moving boundary problem. The efficiency of the method was tested on the two-dimensional heat transfer equation with moving boundary, and its accuracy was examined by the Fourier series method for stability analysis. And the equations are unconditionally stable.

The method suggested in this article has been showed by examining the solidification/melting problem, but it is reasonable to expect that principles can be applied to various systems of different geometries and characteristics.

The results also suggest that the present method, the application of which is easier than many other numerical techniques such as spectral methods, finite-element, heat-balance integral methods, can be applied three or more-dimensional problems.

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AUTHOR'S BIOGRAPHY

Dr. Vildan Gülkaç is working as Associate Prof. at Kocaeli University, science and arts faculty, department of mathematics. Her fields of interest include numerical solutions for free and moving boundary problems.