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Super Edge – Antimagic Total Labeling of Fully Generalized Extended Tree

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Abstract: *An (a,d) edge-antimagic vertex labeling of a graph $G=(V,E)$ with v vertices and e edges is a one-to-one map $\lambda : V(G) \to \{1,2,\dots,v\}$ with the property that $\{w(xy) = \lambda(x) + \lambda(y); xy \in E(G)\}$ is equal to $\{a, a+d, \dots, a+(e-1)d\}$, where a and d are two fixed positive integers. An (a,d) edge-antimagic total labeling of a graph is a one-to-one map $\lambda : V(G) \to \{1,2,\dots,e+v\}$ with the property that $\{w(xy) = \lambda(x) + \lambda(xy) + \lambda(y), xy \in E(G)\}$ is equal to $\{a, a+d, \dots, a+(e-1)d\}$, where a and d are two fixed positive integers. A super (a,d) edge antimagic labeling of a graph is a one-to-one map $\lambda : V(G) \to \{1,2,\dots,e+v\}$ with the property that $\lambda\left(V(G)\right) \to \{1,2,\dots,v\}$. This paper deals with the generalization of the Theorems 2.1 to 2.4 of Javaid* [8] *in terms of the construction of $FGETree(n,r,k)$ and proving the existence of a super (a,d) edge-antimagic total labeling on them for $d \in \{0,1,2\}$*

Keywords: *Super $(a,d) - EAT$ labeling, w-trees, extended w-trees, generalized extended w-trees, fully generalized extended trees.*

AMS subject classification (2010) : 05C78

1. Introduction

Let G be a finite, simple and undirected graph with vertex-set $V(G)$ and edge set $E(G)$.Suppose that $|V(G)| = v$ and $|E(G)| = e$. A gereral reference for graph-theoretic ideas can be seen in [3]. A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a total labeling

Labeling of graphs has its origin in the works of Kotzig and Rosa [4,5]. There are several types of graph labelings. The complete survey of graph labeling can be found in [6].

In this paper, we focus on antimagic total labeling. More details on antimagic total labeling can be found in [1].

Definition 1.1. An (s,d)- edge-antimagic vertex $((s,d) - EAV)$ labeling of a graph G is bijective function $\lambda : V(G) \to \{1,2,\dots,v\}$ such that the set of edge-sums, $E(G)\}$,of all edges in G forms an arithmetic progression $\{s, s+d, \dots, s+(e-1)d\}$ where $s > 0$ and $d \geq 0$ are two fixed integers.

Definition 1.2. An (a,d)- edge-antimagic total $((a,d)– EAT)$ labeling of a graph G is a bijective function $\lambda : V(G) \cup E(G) \to \{1,2,\dots,v+e\}$ such that the set of edge-weights,$\{w(xy) = \lambda(x) + \lambda(xy) + \lambda(y), xy \in E(G)\}$, of all edges in G forms an arithmetic progression $\{a, a+d, .., a+(e-1)d\}$ where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an$(a,d) - EAT$ graph.

Definition 1.3. An$(a,d) - EAT$ labeling λ is called a super (a,d)- edge-antimagic total $(super(a,d) - EAT)$ labeling of G if $\lambda\left(V(G)\right) \to \{1,2,\dots,v\}$. Thus a super $(a,d) - EAT$ graph is a graph that admits a $super(a,d) - EAT$ labeling.

The following proposition presents the relation between a $(s,d) - EAV$ labeling and a super $(a,d) - EAT$ labeling.

Proposition 1.4.[2,7] If a (v,e)- graph G has an $(s,d)-EAV$ labeling and a super $(a,d)-EAT$ labeling then

i. G has a super$(s+v+1,d+1)-EAT$ labeling,

ii. Ghas a super $(s+v+e,d-1)-EAT$ labeling.

2. CONSTRUCTION OF GENERALIZED EXTENDED W-TREES

Definition 2.1. Let n,m,r and k be positive integers. Consider a path P on r vertices as $V(P)=\{c_1,c_2,\dots,c_r\}$ with n hanging leaves ${x_i}^1,{x_i}^2,\dots,{x_i}^n$ (respectively , m leaves $y^1,y^2,\dots,y^m$) from each vertex c_i if $1\le i\le r-1$ (respectively if $i=r$). Consider k copies of such paths $P_1,P_2,\dots,P_k$ with ${y_1}^m,{y_2}^m,\dots,{y_k}^m$ as a last hanging leaf from c_r respectively . A generalized extended w-tree is obtained by joining all the vertices ${y_1}^m,{y_2}^m,\dots,{y_k}^m$ to a further vertex a.

M.Javaid [8] proved the following theorems, for different values of d, related to a $super\ (a,d)-EAT$ labeling of generalized extended w- trees denoted by $GEwt(n,m,r,k)$ under certain conditions on $n,m,r,$ and k.

Theorem 2.1. If $n\ge 1, k\ge 3,$ r even and $m\ge\frac{r}{2}(n+1)k+1$ then $G=GEwt(n,m,r,k)$ admits $super\ (a,0)-EAT$ and $super(a^{'},2)-EAT$ labelings.

Theorem 2.2. If $n\ge 1$, $k\ge 3,$ r even and $m\ge\frac{r}{2}(n+1)k+1$ and v even then $G=GEwt(n,m,$ $r,k)$ then admits $super\ (a^{''},1)-EAT$ labeling.

Theorem 2.3. If $n\ge 1$,$k\ge 3,$ r odd and $m\ \ge\ \frac{r-1}{2}(n+1)k+k+1$ then $G=GEwt(n,m,r,k)$ admits a $super\ (a,0)-EAT$ and $super(a^{'},2)-EAT$ labelings.

Theorem 2.4. If $n\ge 1$,$k\ge 3,$ r odd,$m\ \ge\ \frac{r-1}{2}(n+1)k+k+1$ and v eventhen $G=GEwt(n,m,$ $r,k)$ admits $super(a^{''},1)-EAT$ labeling.

3. CONSTRUCTION OF FULLY GENERALIZED EXTENDED TREE ($\boldsymbol{FGETree(n,r,k)}$)

This section gives the definition of $FGETree(n,r,k)$ which is the ultimate generalization of $GEwt(n,m,r,k)$ defined by M.Javaid [8]

Definition 3.1. Let k be a positive integer. Consider the paths P_i on r_i vertices, $i=1,2,\dots,k$ having $V(P_i)=\{C_{i,1,}C_{i,2},\dots,C_{i,r_i}\}$. Attach the pendant vertices

$$x_{i,1}^1\ ,\ x_{i,1}^2\ ,\ x_{i,1}^3,\ \ \dots\ ,x_{i,1}^{n_{i,}1}\quad \text{on}\ \ c_{i,1}$$
$$x_{i,2}^1\ ,\ x_{i,2}^2\ ,\ x_{i,2}^3,\ \ \dots\ ,x_{i,2}^{n_{i,}1}\quad \text{on}\ \ c_{i,1}$$
$$\vdots$$
$$x_{i,r_i}^1\ ,\ x_{i,r_i}^2\ ,\ x_{i,r_i}^3,\ \ \dots\ ,x_{i,r_i}^{n_{i,}r_i}\quad \text{on}\ \ c_{i,r_i}$$

The Fully Generalized Extended Tree$(n,r,k)\big(FGETree(n,r,k)\big)$ is obtained by joining all the vertices $x_{1,r_1}^{n_1,r_1}, x_{2,r_2}^{n_2,r_2},\dots,x_{k,r_k}^{n_k,r_k}$ to a further vertex a. Here $r=\sum_{i=1}^{k}r_i$ and $n=\sum_{i=1}^{k}\sum_{j=1}^{r_i}n_{i,j}$.Thus $FGETree(n,r,k)$ contains $n+r+1$ vertices and $n+r$ edges. The construction of this tree is given in fig.1 for $k=4$.

Note that the naming of all vertices of the first path starts from the bottom while it is from the top for all the other paths. The naming of the hanging edges in the last vertex of all the paths starts from the bottom which is in reverse order for the second path only . Also the naming of the hanging edges at each vertex (except the last vertex) of all the paths starts from top and form the bottom for the first path. Also note that if any of the suffix or superfix value in $x^{l,m}{}_{i,j}$ or $c_{i,j}$ is ≤ 0, then the λ – value (label) of the corresponding vertex is 0.

Remarks

i. In $FGETree(n,r,k)$ the different paths are of different lengths while they are of the same length in $GEwt(n,m,r,k)$.
ii. In each vertex of the paths, the number of hanging leaves is different while it is the same for all vertices of all paths in $GEwt(n,m,r,k)$.
iii. The number of hanging leaves in the last vertex of each path in $FGETree(n.r.k)$ is minimized which results in the reduction of the edge weight of the tree

Thus, the $FGETree(n,r,k)$ is fully generalized, in all possible ways from $GEwt(n,m,rk)$.

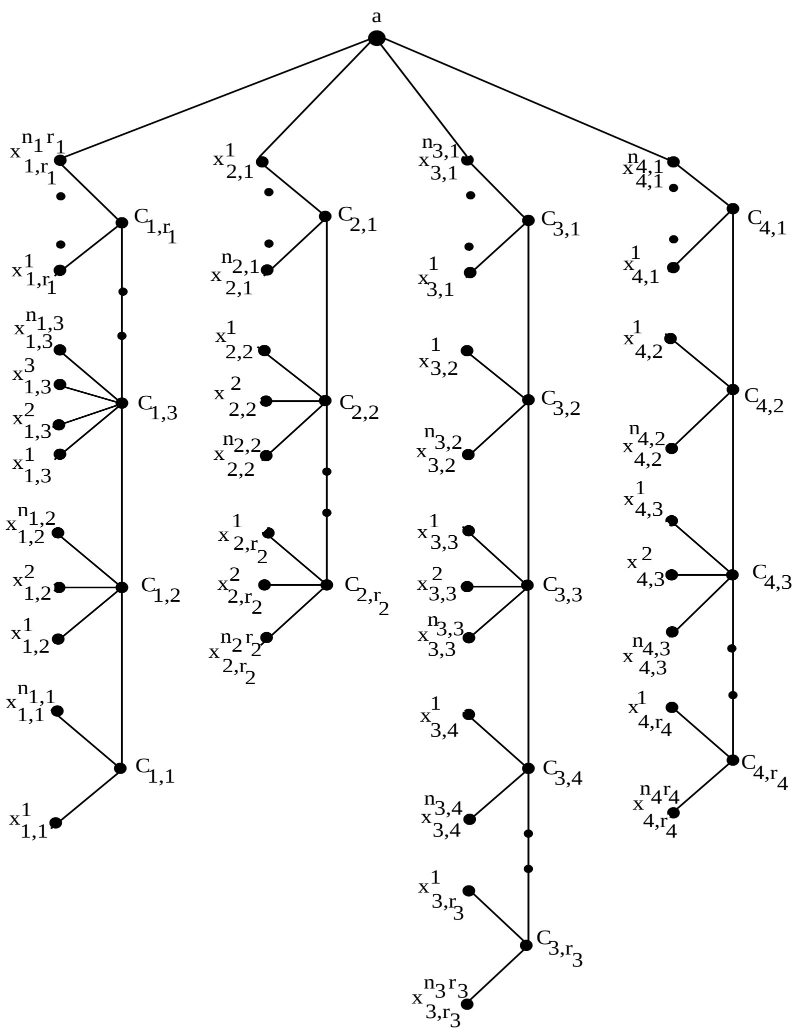

Fig 1. FGETree (n, r, k)

Theorem 3.1. If $n \geq 1,\ k \geq 3, r \geq 1$, then $G = FGETree(n,r,k)$ admits a super $(a,0) - EAT$ and super$(a', 2) - EAT$ labelings.

Proof. Define the vertex-labeling $\lambda : V(G) \to \{1,2,\dots,v\}$ as follows:

For $i = 1, r_i$ is even , then , for $j = 2,4,\dots,r_i$

$$\lambda(c_{i,j}) = \sum_{k=odd}^{j} n_{i,k} + \frac{j}{2}$$

For $i = 1, r_i$ is odd , then, for $j = 1,3,\dots,r_i$

$$\lambda(c_{i,j}) = \sum_{k=even}^{j-1} n_{i,k} + \frac{j+1}{2}$$

$$\lambda(x_{i,j}^{p}) = \begin{cases} \sum_{k=odd}^{j} n_{i,k-2} + \frac{j+1}{2} \\ \text{to} \\ \sum_{k=odd}^{j} n_{i,k} + \frac{j-1}{2} \quad \text{if } i=1,\ j=1,3,\dots,r_i \text{ for } r_i \text{ is even} \\ \text{OR} \\ \sum_{k=even}^{j} n_{i,k-2} + \frac{j+2}{2} \\ \text{to} \\ \sum_{k=even}^{j} n_{i,k} + \frac{j}{2} \quad \text{if } i=1,\ j=2,4,\dots,r_i \text{ for } r_i \text{ is odd} \end{cases}$$

$$\lambda(a) = \lambda(c_1, r_1) + 1, (r_i \text{ is odd or even})$$

$$\lambda(c_{i,j}) = \lambda(a) + \sum_{k=odd}^{j} n_{i,k-1} + \frac{j+1}{2} \text{ if } i=2,\ j=1,3,\dots,r_i$$

$$\lambda(x_{i,j}^{p}) = \begin{cases} \lambda(c_{i,1}) + \sum_{k=even}^{j} n_{i,k-2} + \frac{j}{2} \\ \text{to} \\ \lambda(c_{i,1}) + \sum_{k=even}^{j} n_{i,k} + \frac{j-2}{2} \text{ if } i=2 \text{ and } j=2,4,\dots,r_i \end{cases}$$

For $i = 3,4,\dots,k$, repeat the following equations (1) and (2) alternatively.

$$\lambda(c_{i,j}) = \begin{cases} \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) + \sum_{k=odd}^{j} n_{i,k-1} + \frac{j+1}{2} \text{ if } r_{i-1} \text{ is even}, j=1,3,\dots,r_i \\ \text{OR} \\ \lambda(c_{i-1,r_i-1}) + \sum_{k=odd}^{j} n_{i,k-1} + \frac{j+1}{2} \text{ if } r_{i-1} \text{ is odd}, j=1,3,\dots,r_i \end{cases} \quad \text{.................................(1)}$$

$$\lambda(x_{i,j}^{p}) = \begin{cases} \lambda(c_{i,1}) + \sum_{k=even}^{j} n_{i,k-2} + \frac{j}{2} \\ \text{to} \\ \lambda(c_{i,1}) + \sum_{k=even}^{j} n_{i,k} + \frac{j-2}{2} \quad \text{if } j=2,4,\dots,r_i \end{cases} \quad \text{..(2)}$$

Define for $i = 3,4,\dots,k$

$$n_{i,1} = \begin{cases} \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) - \lambda(a) + 1 \text{ if } r_{i-1} \text{ is even} \\ \lambda(c_{i-1,r_i-1}) - \lambda(a) + 1 \text{ if } r_{i-1} \text{ is odd} \end{cases}$$

Also define

$$R = \begin{cases} \lambda(x_{k,r_k}^{k,r_k}) + 1 & \text{if } r_k \text{ is even} \\ \lambda(c_{k,r_k}) + 1 & \text{if } r_k \text{ is odd} \end{cases}$$

For $i = 1$

$$\lambda(c_{i,j}) = \begin{cases} R + \sum_{k=odd}^{j} n_{i,k-1} + \frac{j-1}{2} & \text{if } r_i \text{ is even, } j = 1,3,\ldots,r_i \\ R + \sum_{k=even}^{j} n_{i,k-1} + \frac{j-2}{2} & \text{if } r_i \text{ is odd, } j = 2,4,\ldots,r_i \end{cases}$$

$$\lambda(x_{i,j}^{p}) = \begin{cases} R + \sum_{k=even}^{j} n_{i,k-2} + \frac{j}{2} & \\ \text{to} & \\ R + \sum_{k=even}^{j} n_{i,k} + \frac{j-2}{2} & \text{if } i = 1,\ j = 2,4,\ldots,r_i\ ,\ r_i \text{ is even} \\ OR & \\ R + \sum_{k=odd}^{j} n_{i,k-2} + \frac{j-1}{2} & \\ \text{to} & \\ R + \sum_{k=odd}^{j} n_{i,k} + \frac{j-3}{2} & \text{if } i = 1,\ j = 1,3,\ldots,r_i\ ,\ r_i \text{ is odd} \end{cases}$$

$$\lambda(x_{i,j}^{p}) = \begin{cases} \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) + \sum_{k=odd}^{j} n_{i,k-2} + \frac{j+1}{2} & \\ \text{to} & \\ \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) + \sum_{k=odd}^{j} n_{i,k} + \frac{j-1}{2} & \text{if } i = 2,\ j = 1,3,\ldots,r_i \end{cases}$$

$$\lambda(c_{i,j}) = \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) + \sum_{k=even}^{j} n_{i,k-1} + \frac{j}{2} \text{ if } i = 2,\ j = 2,4,\ldots,r_i$$

For $i = 3,4,\ldots,k,$ repeat the following equations (3) and (4) alternatively

$$\lambda(x_{i,j}^{p}) = \begin{cases} \lambda(c_{i-1,r_i-1}) + \sum_{k=odd}^{j} n_{i,k-2} + \frac{j+1}{2} & \\ \text{to} & \\ \lambda(c_{i-1,r_i-1}) + \sum_{k=odd}^{j} n_{i,k} + \frac{j-1}{2} & \text{if } r_{i-1} \text{ is even, } j = 1,3,\ldots,r_i \\ \text{OR} & \\ \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) + \sum_{k=odd}^{j} n_{i,k-2} + \frac{j+1}{2} & \\ \text{to} & \\ \lambda(x_{i-1,r_i-1}^{n_{i-1},r_i-1}) + \sum_{k=odd}^{j} n_{i,k} + \frac{j-1}{2} & \text{if } r_{i-1} \text{ is odd, } j = 1,3,\ldots,r_i \end{cases} \quad \ldots\ldots\ldots\ldots(3)$$

$$\lambda(c_{i,j}) = \lambda(x_{i,1}^1) + \sum_{k=odd}^{j} n_{i,k} + \frac{j-2}{2} \text{ if } i = 3,4,\ldots,k \text{ and } j = 2,4,\ldots,r_i \quad \ldots\ldots\ldots\ldots(4)$$

The set of all edge-sums generated by the above labeling forms a consecutive integer sequence $\{s, s+1, s+2, \ldots., s+e\}$ where $s = \lambda(c_{1,1}) + 1$ if r_1 is even and $s = \lambda(x_{1,1}^1) + 1$ if r_1 is odd. Thus, λ is a $(s,1) - EAV$labeling.

Now we extended λ to a super $(a^{'}, 2) - EAT$ labeling using Proposition 1.4. For $i \in \{0, 1, .., e-1\}$,Consider e_i to be the edge having edge-weight $a + d_i$. Label the edge e_i with $f^{'}(e_i) = v + 1 + i$. Combining this new edge label defined by f' and the given vertex labeling definded by λ, we obtain a new $(a^{'}, 2)$ edge-antimagic total labeling for G, where $a^{'} = s + v + 1 = n + r + s + 2$. The set of edge weights under the new labeling f' is equal to $\{(a + v + 1) + (d + i)\, i/i = 0,1, \ldots, e-1\}$.

Similarly λ can be extended to a super $(a, 0) - EAT$ labeling $f^{''}$ instead of f' with magic constant $a = s + v + e = s + 2v - 1$. The label for the edge e_i is given by $f^{''}(e_i) = 2v + e + i - f^{'}(e_i)$, $i \in \{0,1, \ldots, e-1\}$with magic constant $a = v + e + s = 2(n + r) + s + 1$. □

Theorem 3.2. If $n \geq 1, k \geq 3, r \geq 1$, then $G = FGETree(n, r, k)$ admits a super $(a^{''}, 1) - EAT$ labeling.

Proof. Define $V(G), E(G)$ and the vertex-labeling $: V(G) \to \{1,2, \ldots, v\}$ as in theorem 3.1. The set of edge-sums $A = \{a_i : 1 \leq i \leq e\}$, where $a_i = \lambda(c_{1,1}) + i$ or $\lambda(x_{1,1}^1) + i$ according as r_1 is even or odd. Define the set of edge-labels $B = \{b_j : 1 \leq j \leq e\}$ where $b_j = v + j$. Now,the set of edge-weights is defined as $C = \left\{a_{2i-1} + b_{e-i+1} : 1 \leq j \leq \frac{e+1}{2}\right\} \cup \left\{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\right\}$. The set C constitute an arithmetic sequence with $d = 1$ and $a^{''} = s + \frac{3}{2}v = s + \frac{3}{2}(n + r + 1)$. Since all the vertices receive the smallest labels, λ is a super $(a^{''}, 1) - EAT$ labeling.

Thus Theorem 2.1 and Theorem 2.3 are the corollaries of Theorem 3.1 and Theorem 2.2 and Theorem 2.4 are the corollaries of Theorem 3.2

Fig. 2 is an example of a $FGETree$ $(62, 15, 4)$with its first path having odd number of vertices and the fig. 3 is an example of a $FGETree$ $(37, 12, 3)$ with its first path having even number of vertices.

An illustration of the labeling schemes, (where r_1 is odd), presented in Theorem 3.1 is given in fig.2 for $FGETree$ $(62, 15, 4)$ with $v = 78$. As a consequence of the vertex labeling which is formulated in Theorem 3.1 ,fig. 2 gives the set of edge sums which is a sequence of consecutive integers $\{30, 31, .., 106\}$ starting from $s = 30$. Thus, $FGETree(62, 15, 4)$ admits a $(30,1) - EAV$ labling. By the preposition 1.4, we have a super $(a^{'}, 2) - EAT$ labeling with $a^{'} = 109$ and a super $(a, 0) - EAT$ labeling with magic constant $a = 185$. In view of Theorem 3.2, the $FGETree$ $(62, 15, 4)$ admits a super $(a^{''}, 1) - EAT$ labeling with $a^{''} = 147$.

Another illustration of the labeling schemes,(where r_1 is even), is given in fig. 3, for $FGETree(37,12, 3)$with$v = 50$. By the vertex labeling formulated in Theorem 3.1, fig. 3 gives the sequence of consecutive integers $\{23, 24, \ldots, 71\}$ starting from $s = 23$. Hence the $FGETree$ $(37, 12, 3)$ admits a $(23,1) - EAV$ labeling. Thus, by the Preposition 1.4, we have a super $(a^{'}, 2) - EAT$ labeling with $a^{'} = 74$ and a super $(a, 0) - EAT$ labeling with magic constant $a = 122$. Also the $FGETree$ $(37, 12, 3)$ admits a super $(a^{''}, 1) - EAT$ labeling with $a^{''} = 98$, by Theorem 3.2 □

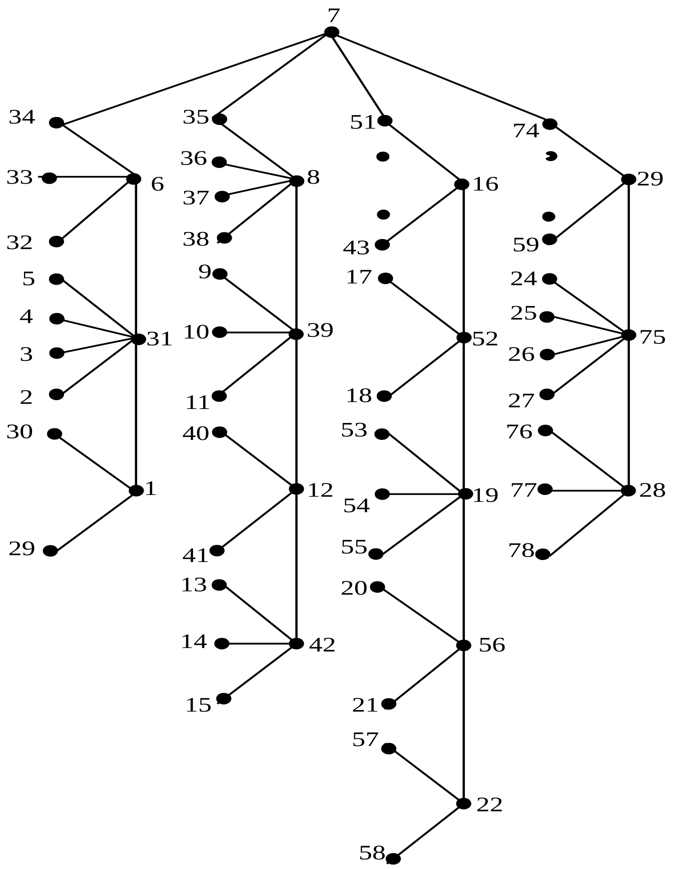

Fig 2. FGETree (62, 15, 4) –its first path having odd number of vertices

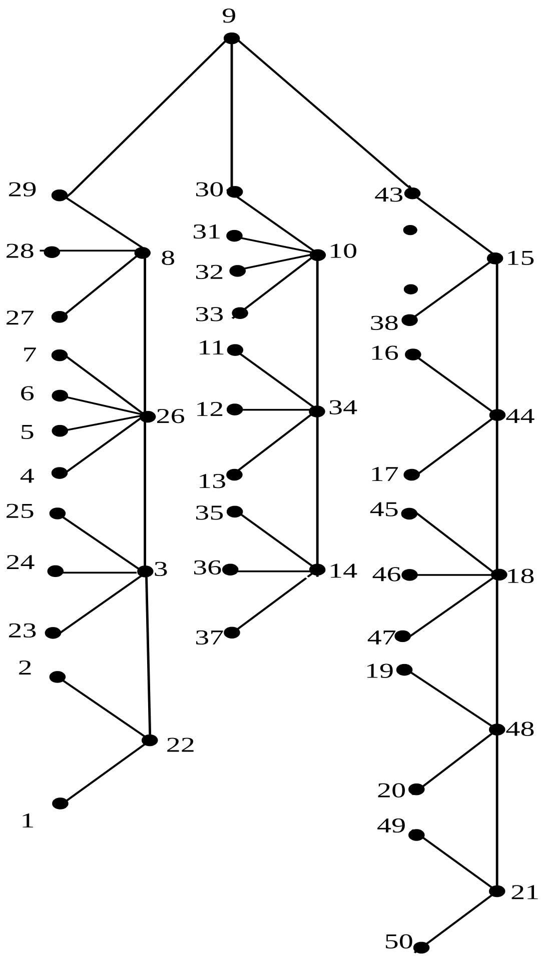

Fig 3. FGETree (37, 12, 3) –its first path having even number of vertices

4 Conclusion

In this paper we presented the definition, construction and the edge-antimagic total labeling of the $FGETree(n, r, k)$. A new problem is to find the $(a, 1)$ edge-antimagic vertex labeling of this $FGETree(n, r, k)$and the related graphs

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