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The Well-posedness and Regularity of An *M/G/l* Queue With Bernoulli Schedules and General Retrial Times

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Abstract*: In this paper, the solution of an* M/G/l *queue with Bernoulli schedules and general retrial times is investigated. By using the method of functional analysis, especially, the linear operator theory and the Co semigroup theory on Banach space, we prove the well-posedness of the system, and show the existence of positive solution.*

Keywords: *Well-posedness;* M/G/1 *retrial queue; Co-semigroup; Dissipative operator; Dispersive operator.*

1. INTRODUCTION

Queueing systems with repeated attempts (retrials) are characterized by the fact that a customer finding all the servers busy upon arrival must leave the service area and repeat his request for service after some random time. Between trials, the blocked customer joins a pool of unsatisfied customers called ”orbit”. Retrial queues have been widely used to model many practical problems in telephone switching systems, telecommunication networks, and computers competing to gain service from a central processing unit. Moreover, retrial queues are used as mathematical models of several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol, and collision avoidance star local area networks. Recent bibliographies on retrial queues can be found in [l-4]. Single server queues with vacations have been studied extensively in the past. A comprehensive survey can be found in [5-7]. These models arise naturally in telecommunications and computer systems, in production and quality control problems, etc. A wide class of policies for governing the vacation mechanism have been discussed in the literature. One of the fundamental features of vacation models is the study of their stochastic decomposition properties. Generally speaking, the stochastic decomposition relates one performance characteristic for the system with vacations to the corresponding one for the same model without vacation. Most of the analysis for retrial queue concerns the exhaustive service schedule [8] and the gated service policy [9]. Consequently, the analysis focuses on characterizing the system performance. Recently, Keilson and Servi [10] have introduced a class of scheduling disciplines, the *M/G/1* Bernoulli service schedule. This is a class of schedules rather than a particular schedule, and provides the opportunity for both performance analysis and system optimization. In this single-server *M/G/*1 Bernoulli vacation model, customers have exponential inter-arrival times and general service times. If the queue is empty after a service completion, then the server becomes inactive, i.e., begins a vacation period, for a duration with a known probability distribution. If the queue is not empty, then another service begins with specified probability p, or a vacation period begins with a probability $q = 1 — p(q > 0)$. At the end of a vacation period, service begins if a customer is present in the queue. Otherwise, the server waits for the first customer to arrive. In this model, each vacation period is an independent identically distributed random variable whose length is independent of the length of the service times. There is extensive literature on many variations of this model. For example, see [11-13]. Retrial queueing systems with general service times and non-exponential retrial time distribution have received little attention. The first work on the *M/G/1* retrial queue with general retrial times is due to Kapyrin [14] who assumed that each customer in orbit generates a stream of repeated attempts that are independent of the customer in orbit and the server state. However, this methodology was found to be incorrect by Falin [2]. Subsequently, Yang et al. [15] have developed an approximation method to obtain the steady- state performance measures for the model of Kapyrin. Fayolle [16] has investigated an *M/M/1* retrial queue where the customers in the retrial group form a queue and only the customer in the head of the queue can request a service to the server after exponentially distributed retrial time with rate a. Farahmand [17] calls this discipline a retrial queue with FCFS orbit. This kind of retrial control policy is well known for the stability of the ALOHA protocol in communication systems [18].

A single-server queue with Bernoulli vacation schedules and general retrial times is studied in Ref.[19]. Kumar et al. present the necessary and sufficient condition for the system to be stable and derive analytical results for the queue length distribution, as well as some performance measures of the system under steady-state condition. The general stochastic decomposition law for *M/G/1* vacation models holds for the present system is showed.

In this paper, we consider a single-server retrial queue with Bernoulli vacation. New customers arrive from outside the system according to a Poisson process with rate λ.We assume that there is no waiting space, and therefore, if an arriving customer finds the server busy or on vacation, the customer leaves the service area and enters a group of blocked customers called 'orbit' in accordance with an FCFS discipline. That is, only the customer at the head of the orbit queue is allowed access to the server. Successive interretrial times of any customer are governed by an arbitrary probability distribution function *A(x)* with corresponding density function a(x). The service times of customers are independent random variables with common distribution function *B(x),* density function *b(x).* The server takes a Bernoulli vacation as described by Keilson and Servi [10], i.e., after each service completion, the server takes a vacation with probability *q*, and with probability *p* = 1 — *q*, he waits for serving the next customer. If the orbit is empty, the server always takes a vacation. At the end of a vacation, the server waits for the customer, if any are in the orbit, or for new customers to arrive. The vacation time *V* has distribution function $V(x)$, density function v(x). Interarrival times, retrial times, service times, and server vacation times are assumed to be mutually independent. From this description, it is clear that either at any service completion epoch or vacation completion epoch, if the server becomes free, in such a case, a possible new arrival and the one (if any) at the head of the orbit queue, compete for service. The function r(x), μ(x), and β(x) are the conditional completion rates (at time *x*) for repeated attempts, for service, and for vacation, respectively, i.e.

$$r(x)=\frac{a(x)}{1-A(x)},\mu(x)=\frac{b(x)}{1-B(x)},\eta(x)=\frac{v(x)}{1-V(x)}.$$

From [19], the system of differential equations associated with the model is following:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \int_0^{+\infty} \eta(x) R_0(x,t) dx, \tag{1.1}$$

$$\frac{\partial P_n(x,t)}{\partial t} + \frac{\partial P_n(x,t)}{\partial x} = -[\lambda + r(x)] P_n(x,t), n = 1,2,..., \tag{1.2}$$

$$\frac{\partial Q_0(x,t)}{\partial t} + \frac{\partial Q_0(x,t)}{\partial x} = -[\lambda + \mu(x)] Q_0(x,t), \tag{1.3}$$

$$\frac{\partial Q_n(x,t)}{\partial t} + \frac{\partial Q_n(x,t)}{\partial x} = -[\lambda + \mu(x)] Q_n(x,t) + \lambda Q_{n-1}(x,t), n = 1,2,..., \tag{1.4}$$

$$\frac{\partial R_0(x,t)}{\partial t} + \frac{\partial R_0(x,t)}{\partial x} = -[\lambda + \eta(x)] R_0(x,t), \tag{1.5}$$

$$\frac{\partial R_n(x,t)}{\partial t} + \frac{\partial R_n(x,t)}{\partial x} = -[\lambda + \eta(x)] R_n(x,t) + \lambda R_{n-1}(x,t), n = 1,2,..., \tag{1.6}$$

And with the boundary conditions

$$P_n(0,t) = p\int_0^{+\infty} \mu(x) Q_n(x,t) dx + \int_0^{+\infty} \eta(x) R_n(x,t) dx, n = 1,2,..., \tag{1.7}$$

$$Q_0(0,t) = \int_0^{+\infty} r(x) P_1(x,t) dx + \lambda P_0(t), \tag{1.8}$$

$$Q_n(0,t) = \int_0^{+\infty} r(x) P_{n+1}(x,t) dx + \lambda \int_0^{+\infty} P_n(x,t) dx, n = 1,2,..., \tag{1.9}$$

$$R_0(0,t) = \int_0^{+\infty} \mu(x) Q_0(x,t) dx, \tag{1.10}$$

$$R_n(0,t) = q\int_0^{+\infty} \mu(x) Q_n(x,t) dx, n = 1,2,.... \tag{1.11}$$

Equations (1.1)—(1.11) should be solved together with the normalizing condition

$$P_0(t) + \int_0^{+\infty} Q_0(x,t) dx + \int_0^{+\infty} R_0(x,t) dx$$
$$+\sum_{n=1}^{\infty}\left[\int_0^{+\infty} P_n(x,t) dx + \int_0^{+\infty} Q_n(x,t) dx + \int_0^{+\infty} R_n(x,t)\right] = 1. \tag{1.12}$$

And an initial conditions

$$P_0(0) = 1; P_n(x,0) = 0, n \ge 1; Q_n(x,0) = 0, r_n(x,0) = 0, n \ge 1;$$

Obviously,the results obtained in Ref. [19] are based on the following hypotheses:

(1) there exists an unique non-negative solution

$p_0(t), P_n(x,t), Q_0(x,t), Q_n(x,t), R_0(x,t), R_n(x,t)$ of this model;

(2) the limits

$$\lim_{t\to+\infty} P_0(t) = P_0, \lim_{t\to+\infty} P_n(x,t) = P_n(x), \lim_{t\to+\infty} Q_0(x,t) = Q_0(x),$$

$$\lim_{t\to+\infty} Q_n(x,t) = Q_n(x), \lim_{t\to+\infty} R_0(x,t) = R_0(x), \lim_{t\to+\infty} R_n(x,t) = R_n(x)$$

exist, where $n = 1, 2, \ldots$.

It is well known that the above hypotheses not always hold and it is necessary to prove the correctness. Base on this motivation, we investigate the well-posedness of the system in the present paper. Furthermore, we show the existence of positive solution.

In this paper we assume that:

(1) λ is a positive constant and $0 < p,q < 1, p + q = 1$;

(2) $r(x), \mu(x), \eta(x)$ are measurable functions and

$$0 < c_1 = \inf_{x\in\mathbb{R}} r(x) < \sup_{x\in\mathbb{R}} r(x) = +\infty, \int_0^{+\infty} e^{-\int_0^x r(s)ds} dx < +\infty; \tag{1.13}$$

$$0 < c_2 = \inf_{x\in\mathbb{R}} \mu(x) < \sup_{x\in\mathbb{R}} \mu(x) = +\infty, \int_0^{+\infty} e^{-\int_0^x \mu(s)ds} dx < +\infty; \tag{1.14}$$

$$0 < c_3 = \inf_{x\in\mathbb{R}} \eta(x) < \sup_{x\in\mathbb{R}} \eta(x) = +\infty, \int_0^{+\infty} e^{-\int_0^x \eta(s)ds} dx < +\infty. \tag{1.15}$$

THE WELL-POSEDNESS OF SYSTEM

In the following, we always denote by $\mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{N}^+$, the real number set, the non-negative real number set, the non-negative integer number set, the positive integer number set, respectively. Let

$$X = \mathbb{R} \times L^1(\mathbb{R}^+ \times \mathbb{N}^+) \times \left(L^1(\mathbb{R}^+ \times \mathbb{N}^+)\right)^2,$$

equipped with the nor*m*

$$\left\|\left(P_0, P_n(x), Q_n(x), R_n(x)\right)\right\| = |P_0| + \sum_{n=1}^{\infty}\left[\|Q_n(x)\|_1 + \|R_n(x)\|_1\right],$$

for $P = \left(P_0, P_n(x), Q_n(x), R_n(x)\right) \in X$. It is easily to see that X is a Banach space.

Now we define rhe following operators: $\mathrm{A} = \mathrm{A}_1 + \mathrm{B}$, where

$$\mathrm{A}_1\begin{pmatrix} P_0 \\ P_n(x) \\ Q_0(x) \\ Q_n(x) \\ R_0(x) \\ R_n(x) \end{pmatrix} = \begin{pmatrix} -\lambda P_0 + \int_0^{+\infty}\eta(x)R_0(x)dx \\ -P_n'(x)-[\lambda+r(x)]P_n(x) \\ -Q_0'(x)-[\lambda+\mu(x)]Q_0(x) \\ -Q_n'(x)-[\lambda+r(x)]Q_n(x) \\ -R_0'(x)-[\lambda+\eta(x)]R_0(x) \\ -R_n'(x)-[\lambda+\eta(x)]R_n(x) \end{pmatrix}, \mathrm{B}\begin{pmatrix} P_0 \\ P_n(x) \\ Q_0(x) \\ Q_n(x) \\ R_0(x) \\ R_n(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda Q_{n-1}(x) \\ 0 \\ \lambda R_{n-1}(x) \end{pmatrix}.$$

The domain of A and A_1 are following:

$D(\mathrm{A}) = D(\mathrm{A}_1) is \left(P_0, P_n(x), Q_0(x), Q_n(x), R_0(x), R_n(x)\right) \in X \ : P_n'(x), r(x)P_n(x) \in L^1(\Omega);$
$Q_0'(x), \mu(x)Q_0(x) \in L^1(\mathbb{R}^+); Q_n'(x), \mu(x)Q_n(x) \in L^1(\Omega); R_0'(x), \eta(x)R_0(x) \in L^1(\mathbb{R}^+);$
$R_n'(x), \eta(x)R_n(x) \in L^1(\Omega); P_n(x), Q_n(x)$ and $R_n(x)$ are all absolutely

continuous, $n = 0,1,2,...; P_n(0) = p\int_0^{+\infty}\mu(x)Q_n(x)dx + \int_0^{+\infty}\eta(x)R_n(x)dx$,

$$Q_0(0) = \int_0^{+\infty} r(x)P_1(x)dx + \lambda P_0, R_0(0) = \int_0^{+\infty}\mu(x)Q_0(x)dx,$$

$$Q_n(0) = \int_0^{+\infty} r(x)P_{n+1}(x)dx + \lambda\int_0^{+\infty} P_n(x)dx, R_n(0) = q\int_0^{+\infty}\mu(x)Q_n(x)dx$$

Then the equation system $(1.1-1.12)$ can be written tne abstract Cauhy problem:

$$\begin{cases} \dfrac{dP(t)}{dt} = \mathrm{A}P(t), \quad t \succ 0 \\ P(0) = \tilde{P}_0. \end{cases} \tag{2.1}$$

Where $P(t) = \left(P_0(t), P_n(x,t), Q_0(x,t), Q_n(x,t), R_0(x,t), R_n(x,t)\right), \tilde{P} = (1,0,0,0,0,0)$.

Theorem 2.1 *The operator A is a linear closed dense defined one in X.*

Proof of Theorem 2.1 is a direct verification,so we obmit the detail.

Let X^* be the dual of X and A_1^* be the dual of A_1,then

$$X^* = \mathbb{R}\times L^\infty\left(\mathbb{R}^+\times N^+\right)\times\left(L^\infty\left(\mathbb{R}^+\times N^+\right)\right)^2,$$

and for any $P = \left(P_0, P_n(x), Q_0(x), Q_n(x), R_0(x), R_n(x)\right) \in D(\mathrm{A}_1)$,

$$Q = \left(p_0, p_n(x), q_0(x), q_n(x), r_0(x), r_n(x)\right) \in X^*$$

We have

$$(\mathrm{A}_1P, Q) = \left[-\lambda P_0 + \int_0^{+\infty}\eta(x)R_0(x)dx\right]p_0 + \int_0^{+\infty}\sum_{n=1}^{\infty}\left\{-P_n'(x) - [\lambda + r(x)]P_n(x)\right\}p_n(x)dx$$

$$\int_0^{+\infty}\left\{-Q_0'(x)-[\lambda+\mu(x)]Q_0(x)\right\}q_0(x)dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-Q_n'(x)-[\lambda+\mu(x)]Q_n(x)\right\}q_n(x)dx$$

$$+\int_0^{+\infty}\left\{-R_0'(x)-[\lambda+\eta(x)]R_0(x)\right\}r_0(x)dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-R_n'(x)-[\lambda+\eta(x)]R_n(x)\right\}r_n(x)dx$$

$$=\lambda[q_0(0)-p_0]P_0+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{p_n'(x)-[\lambda+r(x)]p_n(x)+q_{n-1}(0)r(x)+\lambda q_n(0)\right\}P_n(x)dx$$

$$+\int_0^{+\infty}\left\{q_0'(x)-[\lambda+\mu(x)]q_0(x)+r_0(0)\mu(x)\right\}Q_0(x)dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{q_n'(x)-[\lambda+\mu(x)]q_n(x)+[pp_n(0)+qr_n(0)]\mu(x)\right\}Q_n(x)dx$$

$$+\int_0^{+\infty}\left\{r_0'(x)-[\lambda+\eta(x)]r_0(x)+p_0\eta(x)\right\}R_0(x)dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{r_n'(x)-[\lambda+\eta(x)]r_n(x)+p_n(0)\eta(x)\right\}R_n(x)dx,$$

Where we have used the following equalities

$$\sum_{n=1}^{\infty}\int_0^{+\infty}-P_n'(x)p_n(x)dx=\sum_{n=1}^{\infty}P_n(0)p_n(0)+\sum_{n=1}^{\infty}\int_0^{+\infty}p_n'(x)P_n(x)dx$$

$$=\sum_{n=1}^{\infty}\left[p\int_0^{+\infty}\mu(x)Q_n(x)dx+\int_0^{+\infty}\eta(x)R_n(x)dx\right]p_n(0)+\sum_{n=1}^{\infty}\int_0^{+\infty}p_n'(x)P_n(x)dx,$$

$$\int_0^{+\infty}-Q_0'(x)q_0(x)dx=Q_0(0)q_0(0)+\int_0^{+\infty}q_0'(x)Q_0(x)dx$$

$$=q_0(0)\left[\int_0^{+\infty}r(x)P_1(x)dx+\lambda P_0\right]+\int_0^{+\infty}q_0'(x)Q_0(x)dx,$$

$$\sum_{n=1}^{\infty}\int_0^{+\infty}-Q_n'(x)q_n(x)dx=\sum_{n=1}^{\infty}Q_n(0)q_n(0)+\sum_{n=1}^{\infty}\int_0^{+\infty}q_n'(x)Q_n(x)dx$$

$$=\sum_{n=1}^{\infty}q_n(0)\left[\int_0^{+\infty}r(x)P_{n+1}(x)dx+\lambda\int_0^{+\infty}P_n(x)dx\right]+\sum_{n=1}^{\infty}\int_0^{+\infty}q_n'(x)Q_n(x)dx,$$

$$\int_0^{+\infty}-R_0'(x)r_0(x)dx=R_0(0)r_0(0)+\int_0^{+\infty}r_0'(x)R_0(x)dx$$

$$=r_0(0)\int_0^{+\infty}\mu(x)Q_0(x)dx+\int_0^{+\infty}r_0'(x)R_0(x)dx,$$

$$\sum_{n=1}^{\infty}\int_0^{+\infty}-R_n'(x)r_n(x)dx=\sum_{n=1}^{\infty}R_n(0)r_n(0)+\sum_{n=1}^{\infty}\int_0^{+\infty}r_n'(x)R_n(x)dx$$

$$=\sum_{n=1}^{\infty} r_n(0) q \int_0^{+\infty} \mu(x) Q_n(x) dx + \sum_{n=1}^{\infty} \int_0^{+\infty} r_n'(x) R_n(x) dx.$$

From $(\mathrm{A}P, Q) = (P, \mathrm{A}_1^* Q)$, we obtain

$$\mathrm{A}_1^* \begin{pmatrix} p_0 \\ p_n(x) \\ q_0(x) \\ q_n(x) \\ r_0(x) \\ r_n(x) \end{pmatrix} = \begin{pmatrix} \lambda[q_0(0) - p(0)] \\ p_n'(x) - [\lambda + r(x)] p_n(x) + q_{n-1}(0) r(x) + \lambda q_n(0) \\ q_0'(x) - [\lambda + \mu(x)] q_0(x) + r_0(0)\mu(x) \\ q_n'(x) - [\lambda + \mu(x)] q_n(x) + [p p_n(0) + q r_n(0)]\mu(x) \\ r_0'(x) - [\lambda + \eta(x)] r_0(x) + p_0 \eta(x) \\ r_n'(x) - [\lambda + \eta(x)] r_n(x) + p_n(0)\eta(x) \end{pmatrix},$$

With dpmain

$$D(\mathrm{A}_1^*) = \begin{pmatrix} (p_0, p_n(x), q_0(x), q_n(x), r_0(x), r_n(x)) \in X^* : p_n'(x), r(x) p_n(x) \in L^\infty(\Omega); \\ q_n'(x), \mu(x) q_n(x) \in L^\infty(\Omega); r_n'(x), \eta(x) r_n(x) \in L^\infty(\Omega); \\ q_0'(x), \mu(x) q_0(x) \in L^\infty(\mathbb{R}^+); r_0'(x), \eta(x) r_0(x) \in L^\infty(\mathbb{R}^+). \end{pmatrix}$$

Theorem 2.2 1 is not an eigenvalue of A_1^*.

Proof Let $Q = (p_0, p_n(x), q_0(x), q_n(x), r_0(x), r_n(x)) \in X^*$, such that $\mathrm{A}_1^* Q = Q$, i.e.,

$$\lambda[q_0(0) - p(0)] = p_0, \tag{2.2}$$

$$p_n'(x) - [\lambda + r(x)] p_n(x) + q_{n-1}(0) r(x) + \lambda q_n(0) = p_n(x), \tag{2.3}$$

$$q_0'(x) - [\lambda + \mu(x)] q_0(x) + r_0(0)\mu(x) = q_0(x), \tag{2.4}$$

$$q_n'(x) - [\lambda + \mu(x)] q_n(x) + [p p_n(0) + q r_n(0)]\mu(x) = q_n(x), \tag{2.5}$$

$$r_0'(x) - [\lambda + \eta(x)] r_0(x) + p_0 \eta(x) = r_0(x), \tag{2.6}$$

$$r_n'(x) - [\lambda + \eta(x)] r_n(x) + p_n(0)\eta(x) = r_n(x), \tag{2.7}$$

from $(2.3 - 2.7)$,we get

$$p_n(x) = e^{\int_0^x [1+\lambda+r(s)]ds} \left\{ p_n(0) - \int_0^x [\lambda q_n(0) + q_{n-1}(0) r(u)] e^{-\int_0^u [1+\lambda+r(s)]ds} du \right\}, \tag{2.8}$$

$$q_0(x) = e^{\int_0^x [1+\lambda+\mu(s)]ds} \left\{ q_0(0) - \int_0^x r_0(0)\mu(u) e^{-\int_0^u [1+\lambda+\mu(s)]ds} du \right\}, \tag{2.9}$$

$$q_n(x) = e^{\int_0^x [1+\lambda+\mu(s)]ds} \left\{ q_n(0) - \int_0^x [p p_n(0) + q r_n(0)]\mu(u) e^{-\int_0^u [1+\lambda+\mu(s)]ds} du \right\}, \tag{2.10}$$

$$r_0(x)=e^{\int_0^x[1+\lambda+\eta(s)]ds}\left\{r_0(0)-\int_0^x p_0\eta(u)e^{-\int_0^u[1+\lambda+\eta(s)]ds}du\right\}, \tag{2.11}$$

$$r_n(x)=e^{\int_0^x[1+\lambda+\eta(s)]ds}\left\{r_n(0)-\int_0^x p_n(0)\eta(u)e^{-\int_0^u[1+\lambda+\eta(s)]ds}du\right\}. \tag{2.12}$$

Since $p_n(x),q_0(x),q_n(x),r_0(x),r_n(x)\in L^\infty, n\ge 1$, we have

$$\lim_{x\to+\infty} p_n(x)e^{-\int_0^x[1+\lambda+r(s)]ds}=0,$$

$$\lim_{x\to+\infty} q_0(x)e^{-\int_0^x[1+\lambda+\mu(s)]ds}=0, \lim_{x\to+\infty} q_n(x)e^{-\int_0^x[1+\lambda+\mu(s)]ds}=0,$$

$$\lim_{x\to+\infty} r_0(x)e^{-\int_0^x[1+\lambda+\eta(s)]ds}=0, \lim_{x\to+\infty} r_n(x)e^{-\int_0^x[1+\lambda+\eta(s)]ds}=0,$$

and hence

$$p_n(0)=\int_0^{+\infty}\left[\lambda q_n(0)+q_{n-1}(0)r(u)\right]e^{-\int_0^u[1+\lambda+r(s)]ds}du, \tag{2.13}$$

$$q_0(0)=\int_0^{+\infty} r_0(0)\mu(u)e^{-\int_0^u[1+\lambda+\mu(s)]ds}du, \tag{2.14}$$

$$q_n(0)=\int_0^{+\infty}\left[pp_n(0)+qr_n(0)\right]\mu(u)e^{-\int_0^u[1+\lambda+\mu(s)]ds}du, \tag{2.15}$$

$$r_0(0)=\int_0^{+\infty} p_0\eta(u)e^{-\int_0^u[1+\lambda+\eta(s)]ds}du, \tag{2.16}$$

$$r_n(0)=\int_0^{+\infty} p_n(0)\eta(u)e^{-\int_0^u[1+\lambda+\eta(s)]ds}du. \tag{2.17}$$

Observing that

$$0< a_1=\int_0^{+\infty} r(u)e^{-\int_0^u[1+\lambda+r(s)]ds}du< 1, 0< b_1=\int_0^{+\infty} e^{-\int_0^u[1+\lambda+r(s)]ds}du< 1,$$

$$0< a_2=\int_0^{+\infty} \mu(u)e^{-\int_0^u[1+\lambda+\mu(s)]ds}du< 1, 0< b_2=\int_0^{+\infty} e^{-\int_0^u[1+\lambda+\mu(s)]ds}du< 1,$$

$$0< a_3=\int_0^{+\infty} \eta(u)e^{-\int_0^u[1+\lambda+\eta(s)]ds}du< 1, 0< b_3=\int_0^{+\infty} e^{-\int_0^u[1+\lambda+\eta(s)]ds}du< 1,$$

from (2.14) and (2.16) we have $|q_0(0)| < |r_0(0)| < |p_0|$, and the sign of $q_0(0), r_0(0), p_0$ are same. Combining (2.2) we know that $q_0(0) = r_0(0) = p_0 = 0$. On the other hand, from (2.13), (2.15) and (2.17) we get the linear algebraic equation system of $p_1(0), q_1(0), r_1(0)$:

$$\begin{cases} p_1(0) - b_1\lambda q_1(0) - a_1 = 0 \\ -a_2 p p_1(0) + q_1(0) - a_2 q r_1(0) = 0 \\ -a_3 p_1(0) + r_1(0) = 0 \end{cases} \tag{2.18}$$

its coefficient determinant is $1 + \lambda b_1 a_3 \neq 0$, this means $p_1(0) = q_1(0) = r_1(0) = 0$, By using the method of mathematical induction, we obtain $p_n(0) = q_n(0) = r_n(0) = 0$. Substituting (2.2) and (2.8)-(2.12), we know that $Q = 0$. Therefore, 1 is not a eigenvalue of A_1^*.

We are now in a position to discuss the well-posedness problem of the system (1.1)-(1.12) or shortly (2.1). It is well known that if the system (2.1) is well posed, that is , for each initial $P_0 \in X$, there exists unique a solution $P(t) \in X$, which depends continuously on P_0, then we can define a family of bounded linear operators T(t) : $[0, +\infty) \to L(X)$ by $P(t) = T(t)P_0$.By the property of the solution, the family $\{T(t)\}_{t\geq 0}$ satisfy the following properties:

(1)$T(0) = I, T(t+s) = T(t)T(s)$ and $\|T(t)\| \leq 1$;

(2) For each $P_0, T(t)P_0$ is contiluous function with respect to t,and $\lim_{t\to 0} T(t)P_0 = P_0$;

(3) For each $P_0 \in D(A), AP_0 = \lim_{t\to 0} \frac{T(t)P_0 - P_0}{t}$.

Conversely, if there exists a family of bounded linear operator $\{T(t)\}_{t\geq 0}$ satisfying the three properties above, then *P(t)* = *T(t)Po* is unique a solution to (2.1). In this case, the family of bounded liner operators $\{T(t)\}_{t\geq 0}$ is called strongly continuous semigroup of contraction,

shortly says C_0 semigroup of contraction, and *A* is called the generator of *T*(t). Therefore, the well-posed-ness of the system (2.1) is equivalent to the existence of C_0 semigroup of contraction.

To prove the existence of the semigroup, usually one takes the Laplace transformation for (2.1), this leads to the equation *(sI -A)P* = P_0. Obviously, if $s \in \rho(A)$, then *P*(s) = *R(s,* A)P_0. A question is that the inverse Laplace transform may not exist.

Lumer-Phillips Theorem (see, Ref.[20]) gives a sufficient and necessary condition for *A* is the generator of C_0 semigroup of contraction.

Lemma 2.1. (*Lumer — PhillipsTheorem*) Let X be a Banach space, and X* be the dual space. For each x$\in$ X, the set F(x) = {f $\in$X* | f (x) = $\|x\|^2$ = $\|f\|^2$ }. Let A be a densely defined and closed linear operator form D(A) $\subset$ X to X. In order that A is the generator of C_0 semigroup T(t) of contraction if and only A satisfies the following conditions:

(1) A is dissipative, i.e., for each x$\in$ D(A) there exists a f $\in$F(x) such that $\Re$ (Ax, f) ≤ 0;

(2) There exists a $\rho > 0$ such that R(ρI− A) = X.

We can use the Lumer-Phillips Theorem to show the well-posed-ness of the system (2.1).

Theorem 2.3. (1)A is a dissipative operator on X.

(2) The operator A_i generates a C_0 semigroup of contraction.

Proof First, we show that A_i is a dissipative operator in *X*. In fact, for any $P=(P_0,P_n(x),Q_0(x),Q_n(x),R_0(x),R_n(x))\in D(\mathrm{A}_1)$, , we choose $Q=(p_0,p_n(x),q_0(x),q_n(x),r_0(x),r_n(x))\in X^*$, where

$$p_0=\|P\|\operatorname{sgn}(P_0),\ p_n(x)=\|P\|\operatorname{sgn}(P_n(x)),q_0(x)=\|P\|\operatorname{sgn}(Q_0(x)),$$

$$q_n(x)=\|P\|\operatorname{sgn}(Q_n(x)),r_0(x)=\|P\|\operatorname{sgn}(R_0(x)),r_n(x)=\|P\|\operatorname{sgn}(R_n(x)),$$ Then $(P,Q)=\|P\|\|Q\|$. In addition, we have

$$\begin{aligned}
(\mathrm{A}_1P,Q)=\|P\|\Bigg\{&\left[-\lambda P_0+\int_0^{+\infty}\eta(x)R_0(x)dx\right]\operatorname{sgn}(P_0)\\
&+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-P_n'(x)-[\lambda+r(x)]P_n(x)\right\}\operatorname{sgn}(P_n(x))dx\\
&+\int_0^{+\infty}\left\{-Q_0'(x)-[\lambda+\mu(x)]Q_0(x)\right\}\operatorname{sgn}(Q_0(x))dx\\
&+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-Q_n'(x)-[\lambda+\mu(x)]Q_n(x)\right\}\operatorname{sgn}(Q_n(x))dx\\
&+\int_0^{+\infty}\left\{-R_0'(x)-[\lambda+\eta(x)]R_0(x)\right\}\operatorname{sgn}(R_0(x))dx\\
&+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-R_n'(x)-[\lambda+\eta(x)]R_n(x)\right\}\operatorname{sgn}(R_n(x))dx\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\le\|P\|\Bigg\{&-\lambda|P_0|+\int_0^{+\infty}\eta(x)R_0(x)dx+\sum_{n=1}^{\infty}\left[|P_n(0)|-\int_0^{+\infty}(\lambda+r(x))|P_n(x)|dx\right]\\
&+|Q_0(0)|-\int_0^{+\infty}(\lambda+\mu(x))|Q_0(x)|dx+\sum_{n=1}^{\infty}\left[|Q_n(0)|-\int_0^{+\infty}(\lambda+\mu(x))|Q_n(x)|dx\right]\\
&+|R_0(0)|-\int_0^{+\infty}(\lambda+\eta(x))|R_0(x)|dx+\sum_{n=1}^{\infty}\left[|R_n(0)|-\int_0^{+\infty}(\lambda+\eta(x))|R_n(x)|dx\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\le\|P\|\{&-\lambda|P_0|+\int_0^{+\infty}\eta(X)|R_0(x)|dx+\sum_{n=1}^{+\infty}\left[p\int_0^{+\infty}\mu(X)|Q_n(x)|dx+\int_0^{+\infty}\eta(X)|R_n(x)|dx\right]\\
&-\sum_{n=1}^{+\infty}\int_0^{+\infty}[\lambda+r(x)]|P_n(x)|dx+\left[\int_0^{+\infty}r(x)|P_1(x)|dx+\lambda|P_0|\right]\\
&-\int_0^{+\infty}[\lambda+\mu(x)]|Q_0(x)|dx+\sum_{n=1}^{+\infty}\left[\int_0^{+\infty}r(x)|P_{n+1}(x)|dx+\lambda\int_0^{+\infty}|P_n(x)|dx\right]
\end{aligned}$$

$$-\sum_{n=1}^{\infty}\int_0^{+\infty}[\lambda+\mu(x)]|Q_n(x)|dx+\int_0^{+\infty}\mu(x)|Q_0(x)|dx-\int_0^{+\infty}(\lambda+\eta(x))|R_0(x)|dx$$

$$-\sum_{n=1}^{\infty}\left[q\int_0^{+\infty}\mu(x)|Q_n(x)|dx-\int_0^{+\infty}[\lambda+\eta(x)]|R_n(x)|dx\right]\Bigg\}$$

$$=-\lambda\|P\|\left\{\int_0^{+\infty}[|Q_0(x)|+|R_0(x)|]dx+\sum_{n=1}^{\infty}\int_0^{+\infty}[|Q_n(x)|+|R_n(x)|]dx\right\}=0.$$

Therefore, A_1 is dissipative and hence $R(I - A_1)$ is a closed subspace of *X*. Furthermore, we have $R(I - A_1) = X$. If it is not true, then there exists a $Q \in X^*$, such that for any $F \in R(I - A_1)$, $(F, Q) = 0$. Hence for any $P \in D(A_1)$, $((I - A_1)P, Q) = 0$, i.e., for any $P \in D(A_1)$, $(P, (I - A_1)^* Q) = 0$. Observing that $D(A_1)$ is dense in *X*, thus $A_1^* Q = Q$, this means 1 is a eigenvalue of A_1, which contradicts with Theorem 2.1. Hence $R(I - A_1) = X$. So the Lemma 2.1 asserts that A_1 generates a C_0 semigroup of contraction.

Theorem 2.4. *The operator A generates a* C_0 *semigroup on X. The system (2.1) is well-posed.*

Proof Obviously, *B* is a bounded linear operator on *X*, using the perturbation theory of semigroup ([20]), we know that the operator *A* generates a C_0 semigroup on *X*. Therefore, the system (2.1) is well-possed.

3. THE REGULARITY OF SOLUTION

Since the system (1.1)-(1.12) describes a practical physical state, then an important problem is the existence of positive solution.

In the following, let *X* be a real Banach space. we define a subset X_+ of *X* by

$$X_+=\{(P_0,P_n(x),Q_n(x),R_n(x))\in X \mid P_0\geq 0,P_n(x)\geq 0,Q_n(x)\geq 0,R_n(x)\geq 0\}$$

X_+ is said to be a positive cone. Obviously for each $P \in X$, we can find $P_+, P_- \in X_+$ such that $P=P_+-P_-$. In particular, it holds that $|P|=P_++P_-$ and $\|P\|\leq\||P|\|$.. The Banach space *X* satisfying such properties is said to be Banach lattice. More detail with respect to the positive cone and Banach Lattice can be referred to Ref. [21].

A bounded linear operator *T* is said to be positive operator if $TX_+\subset X_+$.. A C_0 semigroup *T(t)* is said to be a positive semigroup if $TX_+\subset X_+, \forall T\geq 0$.

In order to describe the positive semigroup, we need the following notion.

Definition 3.1. ([21]) *Let X be a Banach lattice,* X_+ *be a positive cone of X and A be a linear operator in X. Denote*

$$G(x)=\left\{\varphi\in X_+^*:(x,\varphi)=\|x_+\|^2=\|\varphi\|^2\right\},$$

if for any $x \in D(A)$*, there exists a* $\varphi\in G(x)$ *such that* $(Ax,\varphi)\leq 0$,*, then A is called the dispersive operator.*

From Ref. [21], we know that the following result is true.

Lemma 3.1. *Let X be a Banach lattice and* A *be a linear closed defined operator on X, then* A *generates a positive contractive semigroup if and only if* A *is a dispersive operator and* $R(I-\mathrm{A})=X$.

Theorem 3.1. *The operator* A *generates a positive* C_0 *contractive semigroup on X.*

Proof: It is well known that X is a Banach lattice. According to Lemma 3.1, it is sufficient to prove that A is a dispersive operator. For any

$$P=\left(P_0,P_n(x),Q_0(x),Q_n(x),R_0(x),R_n(x)\right)\in D(\mathrm{A}),$$

we choose $Q=\|P\|\left([P_0]^+,[P_n(x)]^+,[Q_0(x)]^+,[Q_n(x)]^+,[R_0(x)]^+,[R_n(x)]^+\right)\in X^*$,

where $[P_n(x)]^+=\left([P_1(x)]^+,[P_2(x)]^+,\ldots\right),[Q_n(x)]^+=\left([Q_1(x)]^+,[Q_2(x)]^+,\ldots\right),$

$$[R_n(x)]^+=\left([R_1(x)]^+,[R_2(x)]^+,\ldots\right),$$

And $[a]^+=a$ for $a\succ 0,[a]^+=0$ for $a\le 0.$ Obviously $Q\in G(P)$, and

$$\begin{aligned}
(\mathrm{A}_1P,Q)=\|P\|\Big\{&\left[-\lambda P_0+\int_0^{+\infty}\eta(x)R_0(x)dx\right][P]^+\\
&+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-P_n'(x)-[\lambda+r(x)]P_n(x)\right\}[P_n(x)]^+dx\\
&+\int_0^{+\infty}\left\{-Q_0'(x)-[\lambda+\mu(x)]Q_0(x)\right\}[Q_0(x)]^+dx\\
&+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-Q_n'(x)-[\lambda+\mu(x)]Q_n(x)+\lambda Q_{n-1}(x)\right\}[Q_n(x)]^+dx\\
&+\int_0^{+\infty}\left\{-R_0'(x)-[\lambda+\eta(x)]R_0(x)\right\}[R_0(x)]^+dx\\
&+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-R_n'(x)-[\lambda+\eta(x)]R_n(x)+\lambda R_{n-1}(x)\right\}[R_n(x)]^+dx\Big\}
\end{aligned}$$

$$\begin{aligned}
\le\|P\|\Big\{&-\lambda|P_0|+\int_0^{+\infty}\eta(x)R_0(x)dx+\sum_{n=1}^{\infty}\left[|P_n(0)|-\int_0^{+\infty}(\lambda+r(x))|P_n(x)|dx\right]\\
&+|Q_0(0)|-\int_0^{+\infty}(\lambda+\mu(x))|Q_0(x)|dx+|R_0(0)|-\int_0^{+\infty}(\lambda+\eta(x))|R_0(x)|dx\\
&-\sum_{n=1}^{\infty}\left\{|Q_n(0)|-\int_0^{+\infty}\left[(\lambda+\mu(x))|Q_n(x)|-\lambda|Q_{n-1}(x)|\right]dx\right\}\\
&+\sum_{n=1}^{\infty}\left\{|R_n(0)|-\int_0^{+\infty}\left[(\lambda+\eta(x))|R_n(x)|-\lambda|R_{n-1}(x)|\right]dx\right\}
\end{aligned}$$

$$\le \|P\|\{-\lambda|P_0|+\int_0^{+\infty}\eta(X)|R_0(x)|dx-\sum_{n=1}^{+\infty}\int_0^{+\infty}[\lambda+r(x)]|P_n(x)|dx$$

$$+\sum_{n=1}^{\infty}\left[p\int_0^{+\infty}\mu(x)|Q_n(x)|dx+\int_0^{+\infty}\eta(x)|R_n(x)|dx\right]+\left[\int_0^{+\infty}r(x)|P_1(x)|dx+\lambda|P_0|\right]\}$$

$$-\int_0^{+\infty}[\lambda+\mu(x)]|Q_0(x)|dx+\sum_{n=1}^{+\infty}\left[\int_0^{+\infty}r(x)|P_{n+1}(x)|dx+\lambda\int_0^{+\infty}|P_n(x)|dx\right]$$

$$-\sum_{n=1}^{+\infty}\int_0^{+\infty}(\lambda+\mu(x))|Q_n(x)|dx+\sum_{n=1}^{+\infty}\int_0^{+\infty}\lambda|Q_{n-1}(x)|dx$$

$$+\int_0^{+\infty}\mu(x)|Q_0(x)|dx-\int_0^{+\infty}(\lambda+\eta(x))|R_0(x)|dx$$

$$+\sum_{n=1}^{\infty}\left[q\int_0^{+\infty}\mu(x)|Q_n(x)|dx-\int_0^{+\infty}[\lambda+\eta(x)]|R_n(x)|dx\right]+\sum_{n=1}^{+\infty}\int_0^{+\infty}\lambda|R_{n-1}(x)|dx\Bigg\}=0.$$

The desired result follows from Lemma 3.1.

The following result shows the regularity of the system (2.1).

Theorem 3.2. *Let T(t) be a positive contractive semigroup with generator A, then T(t) satisfies positive conserve property, i.e. , for any* $H_0 \in \in D(A)$ *and* $H_0 > 0$,

$$\|T(t)H_0\|=\|H_0\|,t\ge 0, ||T(t)\ H_0|| = ||H_0||, t\ge 0.$$

Proof Since $H_0 \in\in$ D(A) and $H_0 > 0$, then $T(t)H_0 \in$ D(A) is a classical solution of the system (2.1)

Let

$$P(t)=(P_0(t),P_1(t),P_2(t),R_1(x,y,t),R_2(y,t))=T(t)H_0\succ 0,$$

then P(t) satisfies (1.1)-(1.12). Note that

$$\frac{d}{dt}\|P(t)\|=\frac{d}{dt}\|T(t)H_0\|$$

$$=\frac{dP_0(t)}{dt}+\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{\partial P_n(x,t)}{\partial t}dx+\int_0^{+\infty}\frac{\partial Q_0(x,t)}{\partial t}dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{\partial Q_n(x,t)}{\partial t}dx+\int_0^{+\infty}\frac{\partial R_0(x,t)}{\partial t}dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{\partial R_n(x,t)}{\partial t}dx$$

$$=-\lambda P_0+\int_0^{+\infty}\eta(x)R_0(x,t)dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\{-P_n'(x,t)-[\lambda+r(x)]P_n(x,t)\}dx$$

$$+\int_0^{+\infty}\{-Q_0'(x,t)-[\lambda+\mu(x)]Q_0(x,t)\}dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\{-Q_n'(x,t)-[\lambda+\mu(x)]Q_n(x,t)+\lambda Q_{n-1}(x,t)\}dx$$

$$+\int_0^{+\infty}\left\{-R_0'(x,t)-[\lambda+\eta(x)]R_0(x,t)\right\}dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{-R_n'(x,t)-[\lambda+\eta(x)]R_n(x,t)+\lambda R_{n-1}(x,t)\right\}dx$$

$$=-\lambda P_0+\int_0^{+\infty}\eta(x)R_0(x,t)dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\left\{P_n(0)-\int_0^{+\infty}[\lambda+r(x)]P_n(x,t)dx\right\}$$

$$+Q_0(0)-\int_0^{+\infty}[\lambda+\mu(x)]Q_0(x,t)dx+R_0(0)-\int_0^{+\infty}[\lambda+\mu(x)]R_0(x,t)dx$$

$$+\sum_{n=1}^{\infty}\left\{Q_n(0)-\int_0^{+\infty}[\lambda+\mu(x)]Q_n(x,t)-\lambda Q_{n-1}(x,t)dx\right\}$$

$$+\sum_{n=1}^{\infty}\left\{R_n(0)-\int_0^{+\infty}[\lambda+\eta(x)]R_n(x,t)-\lambda R_{n-1}(x,t)dx\right\}$$

$$=-\lambda P_0+\int_0^{+\infty}\eta(x)R_0(x,t)dx+\sum_{n=1}^{\infty}\int_0^{+\infty}[\lambda+r(x)]P_n(x,t)dx$$

$$+\sum_{n=1}^{\infty}\left[p\int_0^{+\infty}\mu(x)Q_n(x,t)dx+\int_0^{+\infty}\eta(x)R_n(x,t)dx\right]+\int_0^{+\infty}r(x)P_1(x,t)dx+\lambda P_0$$

$$-\int_0^{+\infty}[\lambda+\mu(x)]Q_0(x,t)dx+\sum_{n=1}^{\infty}\left[\int_0^{+\infty}r(x)P_{n+1}(x,t)dx+\lambda\int_0^{+\infty}P_n(x,t)dx\right]$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}[\lambda+\mu(x)]Q_n(x,t)dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\lambda Q_{n-1}(x,t)dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\mu(x)Q_0(x,t)dx-\int_0^{+\infty}[\lambda+\eta(x)]R_0(x,t)dx$$

$$+\sum_{n=1}^{\infty}\left\{q\int_0^{+\infty}\mu(x)Q_n(x,t)dx-\int_0^{+\infty}[\lambda+\eta(x)]R_n(x,t)dx\right\}+\sum_{n=1}^{\infty}\int_0^{+\infty}\lambda R_{n-1}(x,t)dx=0.$$

hence $\|P(t)\|=\|P(0)\|=\|H_0\|$.

4. CONCLUSION

In this paper we investigated an *M/G/*1 retrial queue model with an additional phase of second service and general retrial times. Using functional analysis method we transform the model $(1.1)-(1.12)$ into an abstract operator equation (2.1). The corresponding dynamic operator is *A*. By using the linear operator theory and the C_0 semigroup theory, we obtained the following results:

(1) The operator *A* generates a positive C_0 contractive semigroup on *X*. The system (2.1) is well-posed.

(2) Let $T(t)$ be a positive contractive semigroup with generator A, then $T(t)$ satisfies positive conserve property, i.e., for any $H_0 \in D(A)$ and $H_0 > 0, \|T(t)H_0\| = \|H_0\|, t > 0$.

In practical problems the key point of the issue requires us to analyze completely the system including spectrum of the system operator *A* and finite expansion of solution. From application point of view, the time we can observe the steady state of the system becomes obviously an important index, which is especial important in the investigation of human health problem or recovery. Therefore, after the mathematical modeling for the problem, our task is mainly to solve the following questions：

(1) the system under consideration has a unique nonnegative time-dependent solution;
(2) approximate of solution;
(3) the system has a steady state, and the dynamic solution of the system converges to the steady state.

Obviously, this paper only completed the above question (1).

Let us recall the observation time issue. Let *S* be a reparable system and *P*(t) be the state vector, which describe the probability in the various states. Suppose that the system has a steady state Po. If there is a time ro such that $\|P(t) - P_0\| < 0.25$, $t > T_0$, then it is said that the steady state of *S* is observable at time T_0. Obviously, the observable time T_0 is a more valuable information in application. From the observation time issue we see that it is not only an issue of existence of the solution and steady state but also the quasi-exponential decay issue of the system. How to determine the decay rate of the dynamic solution, however, is hard work, which needs more detail spectral information of the operator determined by the system.

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