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Independent Dominating Polynomial in Graphs

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Abstract: *Let $G = (V, E)$ be a graph of order n. The independent domination polynomial of G is the polynomial $D_i(G, X) = \sum_{j=\gamma_i(G)}^{n} d_i(G, j)\, x^j$, where $d_i(G, j)$ is the number of independent dominating sets of size j. In this paper, we introduce the independent domination polynomial of a graph. The independent domination polynomials of some standard graphs are obtained and some properties of the independent domination polynomial of graphs are established.*

Keywords: *Dominating polynomial, independent dominating polynomial, independent dominating polynomial roots.*

1. Introduction

In a graph $G = (V, E)$, the open neighbourhood of a vertex $v \in V(G)$ is $N(v) = \{x \in V; vx \in E\}$ the set of vertices adjacent to v. The closed neighbourhood is $N[v] = N(v) \cup \{v\}$. The subgraph induced by the set $\langle S \rangle$. A set $S \subseteq V$ is a dominating set if every vertex in V – S is adjacent to a vertex of S and the minimum cardinality of a dominating set is called the domination number of G and is denoted by γ(G). A minimum dominating of a graph G is called a γ- set of G.

A set is independent (or stable) if no two vertices in it are adjacent. The maximum size of the independent set is called independence number of a graph. An independent dominating set of G is a set that is both dominating and independent in G. The independent domination number of G, denoted by $\gamma_i(G)$, is the minimum size of an independent dominating set.

Let G is isomorphic to K_2 with two points' u and v. Then the graph which construct by attach *m* edges in u and *n* edges in υ is called bi-star and denoted by $B(m,n)$. The friendship graph F_n is n triangles in which they have one common vertex. Let G be a star $K_{1,n}$ the graph obtained by subdividing every edge once of the star is called healthy spider and is denoted by sp_n. If we subdivide the $r \leq n = -1$ edges from the star, we will get wounded spider. For more details about the basic definitions which is not appear here, we refer to Harrary [1].

Graph polynomials are powerful and well-developed tools to express graph parameters. Usually graph polynomials are compared to each other by adhoc means allowing deciding whether a newly defined graph polynomial generalizes (or is generalized) by another one. Saeid Alikhani and Peng, Y.H. [4], have introduced the Domination polynomial of a graph. The domination polynomial of a graph G of order n is the polynomial

$D(G, X) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where d(G, i) is the number of dominating sets of G of size i and γ(G) is the domination number of G. This motivated us to introduce the independent domination polynomial of a graph. In this paper, we introduce the independent domination polynomial of a graph. The independent domination polynomial of some standard graphs are obtained, some properties of the independent domination polynomial of a graph are established.

2. Results

Definition 2.1. Let $G = (V, E)$ be a graph of order n with independent domination number the $\gamma_i(G)$ the independent domination polynomial of G is $D_i(G, X) = \sum_{j=\gamma_i(G)}^{n} d_i(G, j)x^j$, *where* $d_i(G, j)$ is the number of independent dominating sets of size j. The roots of the polynomial $D_i(G, X)$ are called the independent dominating roots of G.

Example. Let G = (V, E) be a graph as in Figure 1.

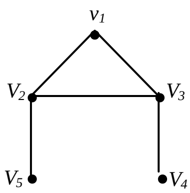


Figure 1

Clearly $\gamma_i(G)$ = 2 and there are only two minimum independent dominating sets {v_3, v_5), {v_2, v_4} and one independent dominating set {v_1, v_4, v_5} of size 3. Hence, $D_i(G, x) = x^2(2+ x)$

Obviously there are two independent dominating roots of G which are 0 and -2.

Observation 2.2. For any graph G = (V, E) The independent Dominating polynomial of G is $D_i(G,x) = \sum_{j=\gamma_i(G)}^{\beta(G)} d_i(G,j)x^j$ where β(G) is the maximum independent number and $\gamma_i(G, J)$, is the number of independent dominating set of size j.

Theorem 2.3. Let G = $G_1 \cup G_2$. Then $D_i(G_j,x) = D_i(G_1,x) D_i(G_2,x)$.

Proof. Any independent dominating set of k vertices in G is constructed by choosing an independent dominating set of j vertices in G_1 (for some $j \in \{\gamma_i(G_1), \gamma_i(G_2), \ldots |V(G_1)|\}$ and the independent dominating set of k – j vertices in G_2. The number of ways of doing this overall j = $\gamma_i(G_1)$, ….., $|V(G_1)|$} is exactly the coefficient of x^k in $D_i(G_1,x) D_i(G_2,x)$.

Theorem 2.4. Let $G \cong \bigcup_{j=1}^{n} G_j$. Then = $D_i(G,x) = \prod_{j=1}^{n} D_i(G_j,x)$.

Proof. We prove this by mathematical induction, the result is true for j=1 is trivial and by Theorem (2.2) for j = 2.

Suppose that $D_i(G,x) = \prod_{j=1}^{n} D_i(G_j,x)$ is satisfy for n=k

i.e., For G = $\bigcup_{j=1}^{k} G_j$

$D_i(G, x) = \prod_{j=1}^{k} D_i(G_j, x)$

Now we have to prove that the result is true for n = k + 1,

So let $G \cong \bigcup_{j=1}^{k+1} G_j \cong \bigcup_{j=1}^{k} G_j \cup G_{k+1}$.

$D_i(G,x) = D_i\left(\bigcup_{j=1}^{k} G_j \cup G_{k+1}\right)$

$= D_i\left(\bigcup_{j=1}^{k} G_j, x\right) D_i(G_{k+1},x)$ by Theorem 2.3

$= \prod_{j=1}^{k} D_i(G_j,x).D_i(G_{k+1},x)$

$= \prod_{j=1}^{k+1} D_i(G_j,x)$.

Hence $D_i(G, x) = \prod_{j=1}^{n} D_i(G_j,x)$.

Corollary 2.5. Let $\overline{k_n}$ be the totally disconnected graph with n vertices. Then $D_i(\overline{k_n},x) = x^n$.

Proof. Let $\overline{k_n}$ is the union of n copies of k_1, if $H \cong K_1$. Then $D_i(H, x) = x$

$D_i(\overline{k_n}) = D_i(K_1,x)\, D_i(K_1,x) \ldots D_i(K_1,x)$

$= \overbrace{x\,x\,x\,x \ldots x}^{n \text{ times}} = x^n$

Theorem 2.6. Let G be a complete graph K_n of n vertices. Then = $D_i(K_n,x)$=n x .

Proof. Let $G \cong K_n$, then there is only n independent dominating set of size one. i.e., $d_i(k_n,1)$=n. Hence $D_i(K_n,x)$ = n x .

Theorem 2.7. Let G is isomorphic to $K_{m,n}$. Then

$$D_i(K_{m,n},x) = x^m(1+x^{n-m})$$

Proof. Let G = (X, Y,E) be the complete bipartite graph with |X| = m and |Y| = n, when $m \leq n$ and $G \cong K_{m,n}$. Then it is obvious that $\gamma_i(K_{m,n})$, = min {m,n}= m. There are only two independent sets which are X and Y. Hence $D_i(K_{m,n}, x) = x^m + x^n = x^m(1+x^{n-m})$

Corollary 2.8.

1) For any star graph $K_{1,n}$, $D_i(K_{1,n}, x) = x\ 1+x^{n-1}$.
2) $d_i(p_2,x)$= 2x.

Theorem 2.9. For any bi-star B (m,n), then $D_i(B(m,n) = x^{m+1} + x^{n+1} + x^{m+n}$

Proof. Let G be a bistar. Then $\gamma_i(B_{m,n})$, = min {m,n} + 1 and it is obvious there are only three independent dominating sets of size n + 1, m + 1, m + n. Hence, $D_i(B_{m,n}) = x^{m+1} + x^{n+1} + x^{m+n}$

Theorem 2.10. For any two graphs G_1 and G_2.

$$D_i(G_1+G_2,x) = D_i(G_1, x) + D_i(G_2,x).$$

Proof. Let G $\cong$ $G_1 + G_2$ and suppose that S_1 is the set of all minimum independent dominating sets in G_1 and S_2 is the set of all minimum independent dominating set in G_2. From the definition of $G_1 + G_2$ it is obvious that every vertex in G_1 will adjacent with each vertex in G_2, so the minimum independent domination $\gamma_i(G_1+G_2) = \min\{\gamma_i(G_1), \gamma_i(G_2)\}$. Any independent dominating set in S_1 will be also independent dominating set of G = $G_1 + G_2$. Similarly any independent dominating set in S_2 will be also independent dominating set of G = $G_1 + G_2$. This means that the independent dominating sets of G = $G_1 + G_2$ is the independent dominating sets of G_1 and G_2.

Therefore,

$$D_i(G, x) = \sum_{j=\gamma_i(G)}^{|V(G)|} d_i(G, j)x^j$$

$$= \sum_{j=\gamma_i(G_1)}^{|V(G_1)|} d_i(G_1, j)x^j + \sum_{j=\gamma_i(G_2)}^{|V(G_2)|} d_i(G_2, j)x^j$$

$$= D_i(G_1, x) + D_i(G_2, x)$$

Theorem 2.11. Let $K_{1,m}$ be a star and G = (V,E) be the spider graph which constructed by subdivision $K_{1,m}$ where m ≥ 3. Then

$$D_i(G,x) = (2^m-1)x^m + x^{m+1}.$$

Proof. Let G = (V, E) be the spider graph, which we get from $K_{1,m}$ by subdivision as in figure 2.

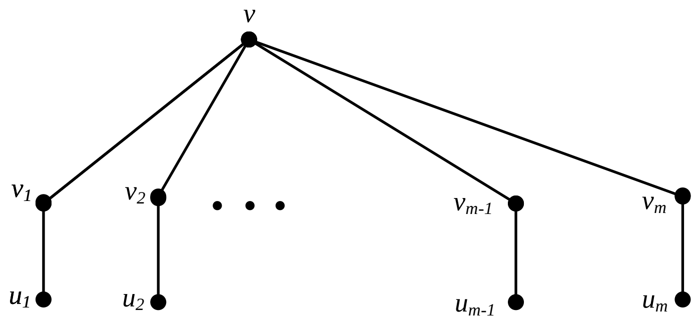


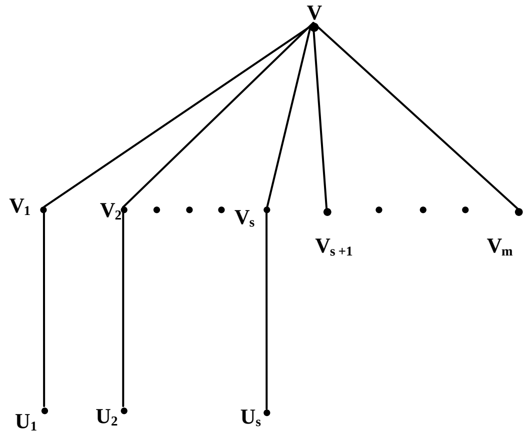
Figure 2. *Healthy spider of* $2n+1$ *vertices*

Observing $\gamma_i(G) = \gamma(G) = m$. Let $A = \{v_1, v_2, v_3, ..., v_{m-1}, v_m\}$ and $B = \{u_1, u_2, u_3, ..., u_{m-1}, u_m\}$ .To find the number of minimum independent dominating sets which of the size m. We can take one vertex from A and m-1 vertices from B, two vertex from A and m-2 vertices from B and so on.

i,e, $d_i(G, m) = \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m-1} + \binom{m}{m} = \sum_{i=0}^{m} \binom{m}{i} - 1 = 2^m - 1$

Also, there is only one independent dominating set of size m+1, which is $\{v, u_1, u_2, \ldots v_m\}$, d_i (G, m + 1) = 1. Hence $D_i(G, x) = x^m(2^m-1+x)$.

Theorem 2.12. Let $K_{1,m}$, where m ≥ 3 be a star and G = (V,E) be the wounded spider graph which constructed by subdivision s ≤ n-1 edges from $K_{1,m}$. $D_i(G,x) = x^{s+1} + (2^s-1)x^m$ where s ≤ m - 1 and m ≥ 3 as in Figure 3.



Proof. Let A = {v_1, v_2,…v_s, v_{s+1},… v_m}, B = {v_1, v_2,…v_s} C = {v_{s+1}, …v_n} D = {u_1, u_2,…u_s}. Now clearly, there is only one minimum independent dominating set of size s+1 i.e. {v,u_1,u_2,….,u_s}. Therefore d_i (G, s + 1) = 1.

To find d_i (G,m) we have the set A and we can take all the vertices of C and one vertex from B and s-1 vertices from D, all the vertices of C and two vertices from B and s-2 vertices from B and so on. Hence $D_i(G,m) = \binom{s}{1} + \binom{s}{2} + \cdots + \binom{s}{s-1} + \binom{s}{s} =$

$2^s - 1, d_i(G,m) = 2^s - 1$

Hence, $D_i(G,x) = x^{s+1} + (2^s - 1)x^m$.

Theorem 2.13. Let G = F_m be a friendship graph with 3_{m+1} vertices. Then $D_i(G,x) = x\,(1+2^m x^{m-1})$.

Proof. Let G be a friendship graph with $3m+1$ vertices as in Figure 4. γ_i(G) = 1. Hence d_i(G, 1) = 1. There is another independent dominating set of size m and there are 2^m ways to make the independent dominating set of size m hence, d_i(G,m) = 2^m. Then $D_i(G, x) = x + 2^m x^m = x\,(1+2^m x^{m+1})$.

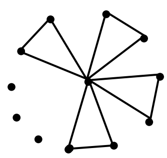


Figure 4. *Friendship graph*

Definition 2.14. A firefly graph $F_{s,t,n-2s-2t-1}$ (s ≥ 0, t ≥ 0 and n – 2s – st – 1 ≥ 0) is a graph of order n that consists of s triangles, t pendent paths of length 2 and n- 2s – 2t – 1 pendant edges sharing a common vertex.

Theorem2.15. Let $G = (V, E)$ be a firefly graph of n vertices, the $D_i(G,x) = x^{t+1} + (2^{s+1})x^{n-s-t-1}$.

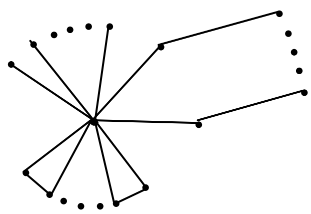


Figure 5. *Firefly graph*

Proof. Let G = (V, E) be a firefly graph $F_{s,t,n-2s-2t-1}$ (s > 0, t > 0, and n – 2s – 2t -1) It is easy to see that γ_i(G) = t + 1 and there is only one minimum independent dominating set which is {V, v_1, v_2,…, v_t}. Hence d_i(G, t+1) = 1. Another independent dominating set of size n – t – s – 1, to find the number of independent dominating set of size $n-t-s-1$, we get $d_i(G, n-t-s-1) = 2(2^s)$. By taking the vertices of triangles alternatively and the triangles of the path of length 2 alternatively. We get, $2(2^s)+2^s(\binom{t}{1}+\binom{t}{2}+\cdots+\binom{t}{t-1}+\binom{t}{t}=$
$2^{s+1}+2^{s+t}-2^{s+1}=2^{s+t}. Hence\, D_i\; G, x\; = x^{t+1}+(2^{s+t})x^{n-t-s-1}$.

3. Conclusion of the Paper

The independent dominating polynomial of a graph is one of the algebraic representation of the graph and quality of any graph representation depend about what information can we get from that presentation about the graph. As this paper is introduced the concept of independent dominating polynomial a still there are a lot of problems can be solved in the future about this concept example is not for limited, the analysis of the roots of the independent dominating polynomial and classification of the graphs we can set the following open problems :

(1) Classification of graphs which has real independent dominating roots.

(2) Classification of graphs which has only three distinct roots.

(3) The summation of the roots of independent domination polynomial of a graph gives us a new parameter of a graph

Recently we work on all these open problems and it will appear recently.

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