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Results on the Commutative Neutrix Convolution Product Involving the Logarithmic Integral li(x^s) and x^r

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Abstract: *The logarithmic integral li(x^s) and its associated functions $li_+(x^s)$ and $li_-(x^s)$ where s = 1, 2,.... are defined as locally summable functions on the real line. The commutative neutrix convolution product of these functions and x^r are evaluated for r = 0, 1, 2,... Further results are also given.*

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Mathematics Subject Classification 2000: *33B10, 46F10*

1. INTRODUCTION

The logarithmic integral li(x), see Abramowitz and Stegun [1] is defined by

$$\mathrm{li}(x)=\begin{cases} \int_0^x \frac{dt}{\ln|t|}, for\ |x|<1 \\ \mathrm{PV}\int_0^x \frac{dt}{\ln t}, \text{for } x>1, \\ \mathrm{PV}\int_0^x \frac{dt}{In|t|}, for\ x<-1 \end{cases}$$

$$=\begin{cases} \int_0^x \frac{dt}{\ln|t|},\ for\ |x|<1 \\ lim_{\varepsilon\to 0^+}[\int_0^{1-\varepsilon} \frac{dt}{\ln t}+\int_{1+\varepsilon}^x \frac{dt}{\ln t}],\ \text{for } x>1, \\ lim_{\varepsilon\to 0^+}[\int_0^{-1+\varepsilon} \frac{dt}{\ln|t|}+\int_{-1-\varepsilon}^x \frac{dt}{\ln|t|}],\ \text{for } x<-1 \end{cases}$$

where PV denotes the Cauchy principal value of the integral, we will use

$$\mathrm{li}(x)=\mathrm{PV}\int_0^x \frac{dt}{In|t|}$$

for all values of x.

The logarithmic integral li(x) was generalized to

$$\mathrm{li}(x^r)=\mathrm{PV}\int_0^{x^r} \frac{dt}{In|t|}$$

and its associated functions $\mathrm{li}_+(x^r)$ and $\mathrm{li}\_(x^r)$ are defined by

$\mathrm{li}_+(x^r)$= H(x) li(x^r), $\mathrm{li}_-(x^r)$= H(-x) li(x^r)

where H(x) denotes Heaviside's function.

It follows that

$$\mathrm{li}(x^r)=\mathrm{PV}\int_0^x \frac{t^{r-1}dt}{In|t|}, \tag{1}$$

see [6]. The distribution $x^{r-1}\ln^{-1}|x|$ is then defined by

$x^{r-1}\ln^{-1}|x|$= [li(x^r)]'

and its associated distributions $x_+^{r-1}\ln^{-1}x_+$ and $x_-^{r-1}\ln^{-1}x_-$ are defined by

$$x_+^{r-1}\ln^{-1} x_+ = H(x)\, x^{r-1}\ln^{-1}|x| = [\mathrm{li}_+(x^r)]',$$
$$x_-^{r-1}\ln^{-1} x_- = H(-x)\, x^{r-1}\ln^{-1}|x| = [\mathrm{li}_-(x^r)]',$$

for r = 1, 2,….

The classical definition of the convolution of two functions f and g is as follows:

Definition1. Let f and g be functions. Then the convolution f*g is defined by

$(f*g)(x)= \int_{-\infty}^{\infty} f(t)g(x-t)dt$

for all points x for which the integral exist.

It follows from Definition 1 that if f * g exists then g*f exists and

f*g= g*f. (2)

Furthermore, if (f*g)' and f*g' (or f'*g) exist, then

(f*g)' = f*g' (or f'*g) (3)

Gel'fand and Shilov [9] extended Definition 1 to define the convolution f*g of two distributions f and g in D', the space of infinitely differentiable functions with compact support.

Definition2. Let f and g be distributions in D'. Then the convolution f*g is defined by the equation

$$\langle (\boldsymbol{f} * \boldsymbol{g})(\boldsymbol{x}), \boldsymbol{\varphi}(x)\rangle = \langle \boldsymbol{f}(\boldsymbol{y}), \langle \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x}+\boldsymbol{y})\rangle\rangle$$

for every $\boldsymbol{\varphi}$ in D, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Note that if f and g are locally summable functions satisfying either of the above conditions and the classical convolution f*g exists, then it is in agreement with Definition 1.1.

The commutative neutrix convolution product is defined in [4] and it works for a large class of pairs of distributions. In that definition, unit-sequences of functions in D are used which allows one to approximate a given distribution by a sequence of distributions of bounded support.

To recall the definition of the commutative neutrix convolution we first let τ be a function in D, see [10], satisfying the the following properties:

i. $\tau(x)= \tau(-x)$,
ii. $0 \le \tau(x) \le 1$,
iii. $\tau(x)= 1$ for $|x| \le \frac{1}{2}$,
iv. $\tau(x)= 0$ for $|x| \ge 1$.

The function τ_n is now defined by

$$\tau_n(x)=\begin{cases} 1, |x| \le n, \\ \tau(n^n x - n^{n+1}), x > n, \\ \tau(n^n x + n^{n+1}), x < -n, \end{cases}$$

for n = 1, 2,….

We have the following definition of the commutative neutrix convolution product.

Definition3. Let f and g be distributions in D' and let $f_n = f\tau_n$ and gn = $g\tau_n$ for n = 1, 2,…. Then the commutative neutrix convolution product f ⊛g is defined as the neutrix limit of the sequence $\{f_n*g_n\}_{n\in N}$, provided the limit h exists in the sense that

$\mathrm{N}-\lim_{n\to\infty}\langle f_n * g_n, \varphi\rangle = \langle h, \varphi\rangle$

for every $\boldsymbol{\varphi}$ in D, where N is the neutrix, see van der Corput [2], having domain N' of positive integers and range N" the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda}\ln^{r-1}n,\ \ln^r n:\ \lambda>0,\ r=1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tend to infinity.

Note that in this definition, the convolution product $f_n * g_n$ is in the sense of Definition 1.1, the distributions f_n and g_n having bounded support since the support of τ_n is contained in the interval $[-n - n^{-n}, n + n^{-n}]$. This neutrix convolution product is also commutative.

It is obvious that any results proved with the original definition hold with the new definition. The following theorems, proved in [4] therefore hold, the first showing that the commutative neutrix convolution product is a generalization of the convolution product. Therefore the idea of a neutrix lies in neglecting certain numerical sequences diverging to ±∞, which makes a wider the class of pairs of distributions f and g for which the product exists. It should be noted that, in general, the definition of a commutative neutrix convolution product depends on the choice of the sequence τ_n as well as the set of negligible sequences.

Theorem1. Let f and g be distributions in D', satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the commutative neutrix convolution product $f \circledast g$ exists and

$$f \circledast g = f*g.$$

Note however that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$, but we do have the following theorem proved in [5].

Theorem2. Let f and g be distributions in D' and suppose that commutative neutrix convolution product $f \circledast g$ exists. If $\underset{n\to\infty}{\mathrm{N}-\lim}\langle (f\tau'_n) * g_n, \varphi\rangle$ exists and equals (h, $\boldsymbol{\varphi}$) for every $\boldsymbol{\varphi}$ in D, then $f' \circledast g$ exists and $(f \circledast g)' = f' \circledast g + h$.

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions $n^s li(n^r)$ and $n^s \ln^{-r} n$, (n > 1) for s = 0, 1, 2,…. and r = 1, 2,….

2. MAIN RESULTS

The following were proved in [6] for r = 0, 1, 2,…., and s = 1, 2,….

$$li_+(x^s)*x_+^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i li_+\left(x^{r+s-i+1}\right), \tag{4}$$

$$x_+^{s-1}ln^{-1}x_+ * x_+^r = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}x^i li_+(x^{r+s-i}) \tag{5}$$

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} \tau_n(t)li(t)(x-t)^r dt = 0 \tag{6}$$

$$\underset{n\to\infty}{\mathrm{N}-\lim}\, li[(x+n)^r] = 0, \tag{7}$$

$$\underset{n\to\infty}{\mathrm{N}-\lim}\, n^r li[(x+n)] = 0 \tag{8}$$

Now we prove the following results.

Theorem 3 The neutrix convolutions $li_+(x^s) \circledast x^r$ exists and

$$li_+(x^s) \circledast x^r = 0, \tag{9}$$

for r = 0, 1, 2,…, and s = 1, 2,….

Proof. Put $[li_+(x^s)]_n = li_+(x^s)\tau_n(x)$ and $[x^r]_n = x^r\tau_n(x)$ for n = 1, 2,…. Since these functions have compact support, the convolution product $[li_+(x^s)]_n*[x^r]_n$ exists by definition 1 and so

$$[li_+(x^s)]_n * [x^r]_n = \int_{-\infty}^{\infty} li_+(t^s)(x-t)^r\tau_n(x-t)\tau_n(t)dt$$

$$= \int_0^n li(t^s)(x-t)^r\tau_n(x-t)dt + \int_n^{n+n^{-n}} li(t^s)(x-t)^r\tau_n(x-t)\tau_n(x-t)\tau_n(t)dt$$

$$= I_1 + I_2. \tag{10}$$

If $0 \le x \le n$, then we have

$I_1 = \int_0^n li(t^s)(x-t)^r \tau_n(x-t)dt$

$= PV\int_0^n (x-t)^r \int_0^t \frac{u^{s-1}}{lnu} du\, dt$

$= PV\int_0^n \frac{u^{s-1}}{lnu} \int_u^n (x-t)^r dt\, du$

$= PV\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1} x^i \binom{r+1}{i} \int_0^n \frac{u^{r+s-i} - n^{r+s-i}}{ln\, u} du$

$= \frac{1}{r+1}\sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i [li(n^{r+s-i+1}) - n^{r+s-i} li(n)].$

Using (7) and (8) we get,

$$N-\lim_{n\to\infty} \int_0^n li(t^s)(x-t)^r \tau_n(x-t)dt = 0. \tag{11}$$

Next, if $-n \le x \le 0$, we have

$I_1 = \int_0^n li(t^s)(x-t)^r \tau_n(x-t)dt$

$= \int_0^{x+n} li(t^s)(x-t)^r dt + \int_{x+n}^{x+n+n^{-n}} li(t^s)(x-t)^r \tau_n(x-t)dt,$

where

$$\int_0^{x+n} li(t^s)(x-t)^r dt = PV\int_0^{x+n} (x-t)^r \int_0^t \frac{u^{s-1}}{\ln|u|} du\, dt$$

$= PV\int_0^{x+n} \frac{u^{s-1}}{\ln u} \int_u^{x+n} (x-t)^r dt\, du$

$= PV\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1} x^i \binom{r+1}{i} \int_0^{x+n} \frac{u^{r-i+s}}{\ln u} du - PV\frac{(-n)^{r+1}}{r+1}\int_0^{x+n} \frac{u^{s-1}}{\ln u} du$

$= \frac{1}{r+1}\sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i\, li[(x+n)^{r+s-i+1}] +$

$- \frac{(-n)^{r+1}}{r+1} li(x+n)^s.$

Using (7) and (8), we have

$$N-\lim_{n\to\infty} \int_0^{x+n} li(t^s)(x-t)^r dt = 0. \tag{12}$$

Furthermore by using (6), we get

$$N-\lim_{n\to\infty} \int_{x+n}^{x+n+n^{-n}} \tau_n(x-t)\, li(t^s)(x-t)^r dt = 0. \tag{13}$$

We have from equations (11), (12) and (13) that

$$N-\lim_{n\to\infty} I_1 = 0. \tag{14}$$

Furthermore, for every fixed x we have

$$\lim_{n\to\infty} I_2 = \lim_{n\to\infty} \int_n^{n+n^{-n}} li(t^s)(x-t)^r \tau_n(x-t)\tau_n(t)dt = 0. \tag{15}$$

Now equation (9) follows from equations (10), (14) and (15), proving the theorem.

Corolary1. The neutrix convolution li_(x^s) ⊛ x^r exists and

$$li\_(x^s) ⊛ x^r = 0, \tag{16}$$

for r = 0, 1, 2,… and s = 1, 2,….

Proof. Equation (16) follows immediately on replacing x by -x in equation(9).

Corolary2. The neutrix convolution li(x^s) ⊛ x^r exists and

$$li(x^s) ⊛ x^r = 0, \tag{17}$$

for r = 0, 1, 2,…, and s = 1, 2,….

Proof. Equation (17) follows on adding equation (9) and (16).

Corolary3. The neutrix convolutions $li_+(x^s) \circledast x^r_-$ and $li_-(x^s) \circledast x^r_+$ exist and

$$li_+(x^s) \circledast x^r_- = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x^i li_+(x^{r+s-i+1}), \tag{18}$$

$$li_+(x^s) \circledast x^r_+ = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i} x^i li_-(x^{r+s-i+1}), \tag{19}$$

for r = 0, 1, 2,…, and s = 1, 2,….

Proof. Equation (18) follows from (4) and (9) by noting that

$li_+(x^s) \circledast x^r = li_+(x^s) \circledast x^r_+ + (-1)^r li_+(x^s) \circledast x^r_-$.

Equation (19) follows by replacing x by -x in equation (18).

Theorem4. The commutative neutrix convolution $x^{s-1}_+ \ln^{-1} x_+ \circledast x^r$ exists and

$$x_+^{s-1}ln^{-1}x_+ \circledast x^r = 0, \tag{20}$$

for r = 0, 1, 2,… and s = 1, 2,….

Proof. Differentiating equation (9) and applying Theorem 2 we get

$$x_+^{s-1}ln^{-1}x_+ \circledast x^r = N - \lim_{n\to\infty}[li_+(x^s)\tau_n'(x)] * (x^r)_n \tag{21}$$

where, on integration by parts we have

$$[li_+(x^s)\tau_n'(x)]*(x^r)_n = \int_n^{n+n^{-n}} li(t^s)(x-t)^r\,\tau_n(x-t)\tau_n'(t)dt$$

$$=-li(n^s)(x-n)^r\tau_n(x-t) - \int_n^{n+n^{-n}} t^{s-1}\,ln^{-1}(t)(x-t)^r\tau_n(x-t)\tau_n(t)dt$$

$$+ r\int_n^{n+n^{-n}} li(t^s)\,(x-t)^{r-1}\tau_n(t)\tau_n(x-t)dt$$

$$+ \int_n^{n+n^{-n}} li(t^s)\,(x-t)^r\tau_n(t)\tau_n'(x-t)dt. \tag{22}$$

Noting that $\tau_n(x - n)$ is either 0 or 1 for large enough n, so

$$N - \lim_{n\to\infty} li(n^s)(x-n)^r\tau_n(x-n) = 0. \tag{23}$$

Also, it is clear that

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} t^{s-1}\,ln^{-1}(t)(x-t)^r\tau_n(t)\tau_n(x-t)dt = 0, \tag{24}$$

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} li\,(t^s)(x-t)^{r-1}\tau_n(t)\tau_n(x-t)dt = 0. \tag{25}$$

Now $\tau'_n(x - t) = 0$ for large enough n and $x \neq 0$, so

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} li\,(t^s)(x-t)^r\tau_n(t)\tau_n'(x-t)dt = 0.$$

If $x = 0$, then

$$\int_n^{n+n^{-n}} li(t^s)\,(x-t)^r\tau_n(t)\tau_n'(-t)dt = \frac{1}{2}li(n^s)(x-n)^r +$$

$$\frac{1}{2}\int_n^{n+n^{-n}}[t^{s-1}ln^{-1}(t)\,(x-t)^r - r li(t^s)(x-t)^{r-1}]\tau_n^2(t)dt. \tag{26}$$

This implies that

$$N - \lim_{n\to\infty}\int_n^{n+n^{-n}} li(t^s)(x-t)^r\,\tau_n(t)\tau_n'(-t)dt = 0 \tag{27}$$

and now equation (20) follows from the equations (22) to (27).

Corolary 4. The neutrix convolution $x^{s-1}\text{ln}\_^{-1}(x) ⊛ x^r$ exists and

$x^{s-1} ln_-^{-1}(x) ⊛ x^r = 0,$ (28)

for r = 0, 1, 2, …, and s = 1, 2,….

Proof. Equations (28) follows by replacing x by - x in equations (20).

Corolary5. The neutrix convolution $x^{s-1}\ln^{-1}|x| ⊛ x^r$ exists and

$x^{s-1}\ln^{-1}|x| ⊛ x^r = 0,$ (29)

for r = 0, 1, 2,…, and s = 1, 2,….

Proof. Equation (29) follows by adding equations (20) and (28).

Corolary6. The neutrix convolutions $x_+^{s-1} ln_+^{-1}(x) ⊛ x_-^r$ and $x_-^{s-1} ln_-^{-1}(x) ⊛ x_+^r$ exist and

$$x_+^{s-1} ln_+^{-1}(x) ⊛ x_-^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x^i li_+\left(x^{r+s-i+1}\right), \quad (30)$$

$$x_-^{s-1} ln_-^{-1}(x) ⊛ x_+^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i} x^i li_+\left(x^{r+s-i+1}\right), \quad (31)$$

for r = 0, 1, 2,… and s = 1, 2,….

Proof. Since

$$x_+^{s-1} ln^{-1}\, x_+ ⊛ x^r = x_+^{s-1} ln^{-1} x_+ ⊛ x_+^r + (-1)^r x_+^{s-1} ln^{-1} x_+ ⊛ x_-^r,$$

equation (30) follows from (5) and (20). Equation (31) follows by replacing x by $-x$ in equation (30).

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