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Strictly Convexity in Quasi 2-pre-Hilbert Spaces

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Abstract: *Using left-hand and right-hand Gateaux derivative of a 2-norm in* [1] *is given functional* $g(x,z)(y)$, *which is generalization of 2-inner product and is used for defining a quasi 2-pre-Hilbert space. Further, in* [1] *is proved that each quasi 2-pre-Hilbert space is smooth. The strictly convexity in quasi 2-pre-Hilbert space is not object of interest in* [1]. *So, in this paper exactly that will be the focus of our interest.*

Keywords: 2-*normed space, smooth space, quasi* 2-*pre-Hilbert space, strictly convex space*

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1. Introduction

Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. Then on $L\times L\times L$ exist the functional

$$N_{+}(x,z)(y)=\lim_{t\to 0^{+}}\frac{\|x+ty,z\|-\|x,z\|}{t},\; N_{-}(x,z)(y)=\lim_{t\to 0^{-}}\frac{\|x+ty,z\|-\|x,z\|}{t}, \tag{1}$$

which are called as left-hand and right-hand Gateaux derivative of the 2-norm $\|\cdot,\cdot\|$ at (x,z) in the direction y, respectively. Therefore, exists the functional

$$g(x,z)(y)=\frac{\|x,z\|}{2}(N_{-}(x,z)(y)+N_{+}(x,z)(y)). \tag{2}$$

The functional $g(x,z)(y)$ is generalization of 2-inner product, and in 2-pre-Hilbert space corresponds to 2-inner product, Theorem 2, [1]. The Theorem 1, [1] proves that in each 2-normed space the following statements are true:

$$g(x,z)(x)=\|x,z\|^{2}, \text{ for every } x,z\in L, \tag{3}$$

$$|g(x,z)(y)|\leq\|x,z\|\cdot\|y,z\|, \text{ for every } x,y,z\in L, \tag{4}$$

$$g(x,z)(x+y)=\|x,z\|^{2}+g(x,z)(y), \text{ for every } x,y,z\in L, \tag{5}$$

$$g(\alpha x,z)(\beta y)=\alpha\beta g(x,z)(y), \text{ for every } x,y,z\in L;\alpha,\beta\in\mathbf{R}, \tag{6}$$

$$\|x,z\|\frac{\|x+\lambda y,z\|-\|x,z\|}{\lambda}\leq g(x,z)(y)\leq\|x,z\|\frac{\|x+ty,z\|-\|x,z\|}{t},\; \lambda<0, t>0, x,y,z\in L. \tag{7}$$

In a 2-pre-Hilbert space hold the parallelepiped equality

$$\|x+y,z\|^{2}+\|x-y,z\|^{2}=2(\|x,z\|^{2}+\|y,z\|^{2}),\; x,y,z\in L \tag{8}$$

and the following one, which is equivalent to the parallelepiped equality

$$\|x+y,z\|^{4}-\|x-y,z\|^{4}=8(\|x,z\|^{2}+\|y,z\|^{2})\cdot(x,y\,|\,z), \tag{9}$$

(Lemma 2, [1]), and thus in 2-normed space the equality

$$\| x+y,z \|^4 - \| x-y,z \|^4 = 8(\| x,z \|^2 \, g(x,z)(y) + \| y,z \|^2 \, g(y,z)(x)), \tag{10}$$

$x, y, z \in L$, generalize (9), i.e. generalize the parallelepiped equality ([1]).

Definition 1 ([1]). A 2-normed space L is called as quasi 2-pre-Hilbert space if the equality (10) holds for every $x, y, z \in L$.

Definition 2 ([2]). A 2-normed space $(L, \| \cdot, \cdot \|)$ is called as smooth if for $x \neq 0$ and $z \notin V(x)$, the 2-norm $\| \cdot, \cdot \|$ is Gateaux differentiable in (x,z) for each direction y.

The quasi 2-pre-Hilbert spaces hold the following Theorem.

Theorem 1 ([1]). Every quasi 2-pre-Hilbert space $(L, \| \cdot, \cdot \|)$ is smooth.

2. ANGLE BETWEEN THE SUBSPACES $V(\{x,z\})$ AND $V(\{y,z\})$

Let $(L,(\cdot,\cdot))$ be a real pre-Hilbert space. Then

$$(x,y \mid z) = \begin{vmatrix} (x,y) & (x,z) \\ (y,z) & (z,z) \end{vmatrix},$$

for every $x, y, z \in L$, defines a standard 2-inner product, and

$$\| x,y \| = (x,x \mid y)^{1/2}$$

defines a standard 2-norm. If $L = \mathbf{R}^3$ with the ordinary scalar product and the vectors $x, y, z \in \mathbf{R}^3$ are not by pairs linearly dependent and $\measuredangle(y,z) = \alpha$, $\measuredangle(z,x) = \beta$ and $\measuredangle(x,y) = \gamma$, that it's easy to compute that

$$\frac{(x,y|z)}{\|x,z\| \cdot \|y,z\|} = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta} = \cos\gamma_1,$$

and γ_1 is an angle between planes parallel to the vectors x,z and y,z. The last, is a real cause of introducing the following definition.

Definition 3. Let $(L,(\cdot,\cdot \mid \cdot))$ be a real 2-pre-Hilbert space, $x, y, z \in L$, the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent and $V(\{x,z\})$ and $V(\{y,z\})$ are subspaces generated by the sets $\{x,z\}$ and $\{y,z\}$, respectively. The angle between the subspaces $V(\{x,z\})$ and $V(\{y,z\})$ is defined by

$$\cos(V(\{x,z\}), V(\{y,z\})) = \frac{(x,y|z)}{\|x,z\| \cdot \|y,z\|}. \tag{11}$$

We already mentioned that the functional $g(x,z)(y)$ is generalization of 2-inner product, and exist in each 2-normed space. Using this and the inequality (4), follows that in 2-normed space, the following definition of an angle between the subsets $V(\{x,z\})$ and $V(\{y,z\})$ is regular.

Definition 4. Let $(L, \| \cdot, \cdot \|)$ be a real 2-normed space, $x, y, z \in L$, the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent and $V(\{x,z\})$ and $V(\{y,z\})$ are subspaces generated by the sets $\{x,z\}$ and $\{y,z\}$, respectively. The angle between the subspaces $V(\{x,z\})$ and $V(\{y,z\})$ is defined by

$$\cos(V(\{x,z\}), V(\{y,z\})) = \frac{g(x,z)(y) + g(y,z)(x)}{2\|x,z\| \cdot \|y,z\|}. \tag{12}$$

Example 1. In [3] is proved that in the set of bounded sequences of real numbers l^∞ with

$$\| x,y \| = \sup_{\substack{i,j \in \mathbf{N} \\ i<j}} \left| \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \right|, \quad x = (x_i)_{i=1}^{\infty}, \; y = (y_i)_{i=1}^{\infty} \in l^\infty$$

is defined 2-norm. It means that $(l^{\infty},\|\cdot,\cdot\|)$ is real 2-normed space. It is easy to check the validity of

$$\|x,z\|=\|y,z\|=1,\ N_{+}(x,z)(y)=1=N_{-}(x,z)(y),$$

$$N_{+}(y,z)(x)=1=N_{-}(y,z)(x) \text{ and } g(x,z)(y)=1=g(y,z)(x)$$

for the vectors

$$x=(1-\tfrac{1}{2},1-\tfrac{1}{2^2},...,1-\tfrac{1}{2^n},...),\ y=(0,1-\tfrac{1}{2},1-\tfrac{1}{2^2},...,1-\tfrac{1}{2^{n-1}},...) \text{ and } z=(1,0,0,...,0,...),$$

and using (12) we get

$$\cos(V(\{x,z\}),V(\{y,z\}))=\tfrac{g(x,z)(y)+g(y,z)(x)}{2\|x,z\|\cdot\|y,z\|}=1.$$

3. CONVEXITY IN QUASI 2-PRE-HILBERT SPACE

Definition 5 ([4]). Let L be a real vector space with $\dim L>1$ and the function $[\cdot,\cdot|\cdot]:L^3\to\mathbf{R}$ is such that

1) $[x,x|y]>0$, if the set $\{x,y\}$ is linearly independent,
2) $[x,x|y]=[y,y|x]$, for each $x,y\in L$,
3) $[\lambda x,y|z]=\lambda[x,y|z]$, for each $x,y,z\in L$ and for each $\lambda\in\mathbf{R}$,
4) $[x+x',y|z]=[x,y|z]+[x',y|z]$, for each $x',x,y,z\in L$,
5) $|[x,y|z]|^2\le[x,x|z]\cdot[y,y|z]$, for each $x,y,z\in L$.

The function $[\cdot,\cdot|\cdot]$ is called as 2-semi-inner product and the pair $(L,[\cdot,\cdot|\cdot])$ is called 2-semi-inner product space.

Theorem 2. If 2-normed space L is smooth, then the functional $g(x,z)(y)$ is linear in terms of y and further more

$$[y,x|z]=g(x,z)(y), \tag{13}$$

for each $x,y,z\in L$ defines 2-semi-inner product.

Proof. Let L be smooth. Then the 2-norm is Gateaux differentiable in (x,z) in each direction y, i.e.

$$N_{-}(x,z)(y)=N_{+}(x,z)(y)=N(x,z)(y)$$

and using (2), follow

$$g(x,z)(y)=\|x,z\|N(x,z)(y). \tag{14}$$

Further, if the set $\{x,y\}$ is linearly independent, the theorems 2.2 and 2.4, [5] imply followings

$$[y,y|x]=g(y,x)(y)=\|y,x\|N(y,x)(y)=\|y,x\|^2>0, \text{ and } [y,y|x]=\|y,x\|^2=\|x,y\|^2=[x,x|y]$$

i.e. hold the axioms 1) and 2) of definition 5. The theorem 2.4, [5] and the mentioned above imply

$$[\lambda y,x|z]=\|x,z\|N(x,z)(\lambda y)=\lambda\|x,z\|N(x,z)(y)=\lambda[y,x|z]$$

$$\begin{aligned}[y+y',x|z]&=g(x,z)(y+y')=\|x,z\|N(x,z)(y+y')\\&=\|x,z\|(N(x,z)(y)+N(x,z)(y'))\\&=g(x,z)(y)+g(x,z)(y')=[y,x|z]+[y',x|z],\end{aligned}$$

$$|[y,x\,|\,z]|^2 = |g(x,z)(y)|^2 = \|x,z\|^2 \cdot |N(x,z)(y)|^2$$
$$\leq \|x,z\|^2 \cdot \|y,z\|^2 = [x,x\,|\,z]\cdot[y,y\,|\,z],$$

i.e. hold the axioms 3), 4) and 5) of definition 5. This actually means that (13) defines a 2-semi-inner product on L.

The following Theorem about 2-semi-inner product is fully true:

Theorem 3 ([6]). If L be a space with 2-semi-inner product $[\cdot,\cdot\,|\,\cdot]$, then L is 2-normed space in which 2-norm is defined by

$$\|x,y\| = [x,x\,|\,y]^{1/2},\; x,y \in L. \tag{15}$$

If L be a 2-normed space, then in L may be introduced 2-semi-inner product $[\cdot,\cdot\,|\,\cdot]$ (in generally, not unique), compatible with 2-norm, i.e. such that (15) holds.

Definition 6 ([7]). Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space, $x,y \in L$ are non-zero elements and $V(x,y)$ denotes the subspace of L generated by the vectors x and y. Space L is strictly convex if $\|x,z\| = \|y,z\| = \|\frac{x+y}{2},z\| = 1$ and $z \notin V(x,y)$, for $x,y,z \in L$, imply $x = y$.

More characterizations of strictly convex 2-normed spaces are given in papers [2] and [8] – [21]. In paper [6] is given a following characterization of strictly convexity into a space with 2-semi-inner product.

Theorem 4. Let L be a 2-normed space and $[\cdot,\cdot\,|\,\cdot]$ be 2-semi-inner product compatible with 2-norm. Then L is strictly convex if and only if

$$[x,y\,|\,z] = \|x,z\|\cdot\|y,z\|,\; z \notin V(x,y), \text{ implies } y = \lambda x, \text{ for some } \lambda > 0.$$

Theorem 5. Let L be a smooth 2-normed space. Then L is strictly convex if and only if

$$\cos(V(\{x,z\}),V(\{y,z\})) = 1,\; z \notin V(x,y), \text{ implies } y = \lambda x, \text{ for some } \lambda > 0. \tag{16}$$

Proof. Let L be a smooth and strictly convex space. If

$$\cos(V(\{x,z\}),V(\{y,z\})) = 1,\; z \notin V(x,y).$$

then

$$g(x,z)(y) + g(y,z)(x) = 2\|x,z\|\cdot\|y,z\|,\; z \notin V(x,y)$$

and by (4) we get following

$$g(x,z)(y) = g(y,z)(x) = \|x,z\|\cdot\|y,z\|,\; z \notin V(x,y). \tag{17}$$

But, L is smooth, so by Theorem 2, we get $g(y,z)(x)$ is 2-semi-inner product compatible with 2-norm in which holds (17). Finally, by Theorem (4), we get $y = \lambda x$, for some $\lambda > 0$.

Conversely, let L be a smooth and the condition (16) be satisfied. By Theorem 2, we get that (13) defines a 2-semi-inner product compatible with the 2-norm. Let suppose,

$$g(x,z)(y) = \|x,z\|\cdot\|y,z\|,\; z \notin V(x,y). \tag{18}$$

Let $z \notin V(x,y)$ and $L_z = L/V(z)$ and $\|x_z\|_z = \|x,z\|$. Then, $(L_z,\|\cdot\|_z)$ is normed space, ([9]). Moreover, the space L is smooth, and so,

$$N(x,z)(y) = \lim_{t\to 0}\frac{\|x+ty,z\|-\|x,z\|}{t} = \lim_{t\to 0}\frac{\|x_z+ty_z\|-\|x_z\|}{t} = N(x_z,y_z)$$

It means, the space L_z is smooth and by the equality (14) we get, the functional

$$g_z(x_z,y_z) = \|x_z\|_z\, N(x_z,y_z),$$

is linear in terms of y_z. Now, the equality (18) implies

$$g_z(x_z,y_z)=\|x_z\|_z\|y_z\|_z,\text{ for } x_z\neq 0_z,\ y_z\neq 0_z, \tag{19}$$

So, using Theorem 3, p.p. 25 [22] we get $y_z \perp_B g_z(x_z,\cdot)$. The last conclusion used in Theorem 2 [5] in fact means that $g_z(y_z,u_z)=0$, for each $u_z\in g_z(x_z,\cdot)$. Let suppose that

$$y_z=\lambda x_z+u_z,\ \lambda\in\mathbf{R},\ \ u_z\in g_z(x_z,\cdot).$$

Then,

$$\begin{aligned}\|y_z\|_z^2&=g_z(y_z,y_z)=g_z(y_z,\lambda x_z+u_z)\\&=\lambda g_z(y_z,x_z)+g_z(y_z,u_z),\end{aligned}$$

i.e.

$$\lambda g_z(y_z,x_z)=\|y_z\|_z^2. \tag{20}$$

But,

$$\begin{aligned}g_z(x_z,y_z)&=g_z(x_z,\lambda x_z+u_z)=\lambda g_z(x_z,x_z)+g_z(x_z,u_z)\\&=\lambda g_z(x_z,x_z)=\lambda\|x_z\|_z^2,\end{aligned}$$

Therefore the last equality and the equality (19) imply

$$\|y_z\|_z=\lambda\|x_z\|_z,\ \lambda>0.$$

Finally, by the last equality and the equality (20) we get the following

$$g_z(y_z,x_z)=\|y_z\|_z\|x_z\|_z.$$

Now, the arbitrarily of $z\notin V(x,y)$ and the last equality, imply

$$g(y,z)(x)=\|x,z\|\cdot\|y,z\|,\ z\notin V(x,y). \tag{21}$$

By equalities (18) and (20) we get

$$\cos(V(\{x,z\}),V(\{y,z\}))=1,\ z\notin V(x,y),$$

and by assumption, $y=\lambda x$, for some $\lambda>0$.

Finally, the assumption

$$g(x,z)(y)=\|x,z\|\cdot\|y,z\|,\ z\notin V(x,y)$$

implies $y=\lambda x$, for some $\lambda>0$. And, by Theorem 4 follows L is strictly convex.

Corollary 1. If L be a quasi 2-pre-Hilbert space, then L is strictly convex.

Proof. Let L be a quasi 2-pre-Hilbert space, i.e. let be satisfied the condition (10). If firstly, we use the equality (10), and then (6) we get

$$\begin{aligned}16-\left\|\tfrac{x}{\|x,z\|}-\tfrac{y}{\|y,z\|},z\right\|^4&\geq\left\|\tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z\right\|^4-\left\|\tfrac{x}{\|x,z\|}-\tfrac{y}{\|y,z\|},z\right\|^4\\&=\frac{\|x\|y,z\|+y\|x,z\|,z\|^4-\|x\|y,z\|-y\|x,z\|,z\|^4}{\|x,z\|^4\|y,z\|^4}\\&=\frac{8(\|x\|y,z\|,z\|^2g(x\|y,z\|,z)(y\|x,z\|)+\|y\|x,z\|,z\|^2g(y\|x,z\|,z)(x\|y,z\|))}{\|x,z\|^4\|y,z\|^4}\\&=\frac{8\|x,z\|^3\|y,z\|^3(g(x,z)(y)+g(y,z)(x))}{\|x,z\|^4\|y,z\|^4}\\&=16\cos(V(\{x,z\}),V(\{y,z\})),\end{aligned}$$

i.e.

$$16\cos(V(\{x,z\}),V(\{y,z\}))\leq 16-\left\|\tfrac{x}{\|x,z\|}-\tfrac{y}{\|y,z\|},z\right\|^4. \tag{20}$$

Further, if

$$\cos(V(\{x,z\}),V(\{y,z\}))=1,\ z\notin V(x,y),$$

by (20) we get

$$\| \frac{x}{\|x,z\|}-\frac{y}{\|y,z\|},z\|^4\leq 0,\ z\notin V(x,y).$$

Thus, $y=\frac{\|y,z\|}{\|x,z\|}x$, i.e. the condition (16) holds. But, by Theorem 1, the space L is smooth, and so, theorem 5 implies that the space is strictly convex.

4. CONCLUSION

By corollary 1, we proved each quasi 2-pre-Hilbert space is strictly convex, and in Theorem 5 we gave a criteria a smooth 2-normed space to be a strictly convex. It is naturally to state the question about necessary and sufficient conditions a quasi 2-pre-Hilbert space to be continuously convex, and to be considered additional properties of quasi 2-pre-Hibert spaces.

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