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On e-labeling of Graphs

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Abstract: *Let $G(V,E)$ be a graph of order n and size m. A e-labeling of G is a one-to-one function $f:V(G)\to\{0,1,2,...,m\}$ that induces a labeling $f^+:E(G)\to\{1,2,3,...,m^2\}$ of the edges of G defined by $f^+(uv)=|\,[f(u)]^2-[f(v)]^2\,|$ for every edge uv of G. The value of a e-labeling is denoted by e-val$(f)=\sum_{uv\in E} f^+(uv)$. The maximum value of a e-labeling of G is defined by e-val$_{max}(G)$ = max{ e-val (f) : f is a e-labeling of G }, while the minimum value of a e-labeling of G is defined by e-val$_{min}(G)$ = min{e-val(f) : f is a e-labeling of G }. In this paper, we investigate the e-val$_{min}(G)$ and e-val$_{max}(G)$ of path P_n, cycle C_n, star $K_{1,n-1}$ and wheel W_{n-1}.*

Keywords: *e-labeling , maximum value , minimum value.*

Mathematical subject classification (2010) 05C78

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V,E)$ be a graph with $p=|V(G)|$ vertices and $q=|E(G)|$ edges. Graph labeling , where the vertices are assigned values subject to certain conditions. By graph labeling we mean the vertices and edges are assigned real values or subsets of a set are subject to certain conditions. A detailed survey of graph labeling can be found in [3]. There are different types of labelings such as graceful labeling , harmonious labeling, γ-labeling, skolem labeling and mean labeling etc applied to various classes of graphs. In this paper we introduce a new labeling .We use the following definitions in the subsequent sections.

Definition 1.1: Let $G(V,E)$ be a graph of order n and size m. A e-labeling of G is a one-to-one function $f:V(G)\to\{0,1,...,m\}$ that induces a labeling $f^+:E(G)\to\{1,2,3,...,m^2\}$ of the edges of G defined by $f^+(uv)=|\,[f(u)]^2-[f(v)]^2\,|$ for every edge uv of G. The value of a e-labeling is denoted by e-val$(f)=\sum_{uv\in E} f^+(uv)$. The maximum value of a e-labeling of G is defined by e-val$_{max}(G)$ = max{ e-val (f) : f is a e-labeling of G }, while the minimum value of a e-labeling of G is defined by e-val$_{min}(G)$ = min{e-val(f) : f is a e-labeling of G }.

Definition 1.1 [1]: For a graph G of order n and size m, a γ-labeling of G is a one-to-one function $f:V(G)\to\{0,1,...,m\}$ that induces a labeling $f':E(G)\to\{1,2,3,...,m\}$ of the edges of G defined by $f'(uv)=|\,[f(u)]-[f(v)]|$ for each edge uv of G. Each γ-labeling f of a graph G of order n and size m is assigned a value denoted by val(f) and defined by val(f) = $\sum_{uv\in E} f'(uv)$. The maximum value of a γ-labeling of graph G is defined by val$_{max}$(G) = $\max\{$ val(f) : f is a γ-labeling of $G\}$, while the minimum value of a γ-labeling of G is defined by val$_{min}$(G)= $\min\{$ val(f) : f is a γ-labeling of $G\}$.

Definition 1.3 [3]: The wheel W_n is obtained by joining all nodes of cycle C_n to a further node called the center, and contains $n+1$ nodes and $2n$ edges.

Definition 1.4 [3]: A complete bipartite graph $K_{1,n}$ is called a star and it has $n+1$ vertices and n edges.

2. MAIN RESULTS

Theorem 2.1: Let P_n be a path of order n. Then e-val$_{min}(P_n) = (n-1)^2$.

Proof: Let P_n be a path with n vertices and $n-1$ edges. Let $V(P_n) = \{v_i : 1 \le i \le n\}$.

Let $E(P_n) = \{v_i v_{i+1} : 1 \le i \le n-1\}$. Then size $m = n-1$. Let f be a e-labeling of P_n.

The n vertices of the path P_n are labelled by $f(v_i) = i-1 \;\; if \;\; 1 \le i \le n$. And f induces that $f^+ : E(P_n) \to \{1,2,3,...,m^2\}$ by $f^+(uv) = |\,[f(u)]^2 - [f(v)]^2\,|$ for every edge uv of path P_n. Then the induced edge label are as follows: $f^+(v_i v_{i+1}) = 2i-1 \;\; if \;\; 1 \le i \le n-1$.

The minimum label of edges of P_n are $\{1,3,5,...,2n-3\}$.

Then e-val$_{min}(P_n) = \sum f^+(uv) = 1+3+5+...+(2n-3) = (n-1)^2$.

Theorem 2.2: Let P_n be a path graph of order n. For every odd integer $n \ge 7$,

Then e-val$_{max}(P_n) = \dfrac{(n^3 - n^2 - 3n + 1)}{2}$ if $n = 4k-1$, $k \ge 2$ and

$= \dfrac{(n^3 - n^2 - 3n + 5)}{2}$ if $n = 4k+1$, $k \ge 2$.

Proof: Let P_n be a path with $n \ge 7$ vertices and $n-1$ edges and where n is an odd integer.

Let $V(P_n) = \{v_i : 1 \le i \le n\}$. Let $E(P_n) = \{v_i v_{i+1} : 1 \le i \le n-1\}$. Then size $m = n-1$.

Let f be a e-labeling of P_n. Let f^+ be the induced edge labeling of f.

Case (i) [$n = 4k-1$, $k \ge 2$]

The n vertices of the path P_n are labelled by $f(v_{\frac{n+1}{2}}) = 0$;

$$f(v_{2i-1}) = \frac{n-3+4i}{2} \;\; if \;\; 1 \le i \le \frac{(n+1)}{4};\;\; f(v_{2i}) = \frac{n+1-4i}{2} \;\; if \;\; 1 \le i \le \frac{(n-3)}{4};$$

$$f(v_{\frac{n-1+4i}{2}}) = n-2i \;\; if \;\; 1 \le i \le \frac{(n+1)}{4};\;\; f(v_{\frac{n+1+4i}{2}}) = 2i-1 \;\; if \;\; 1 \le i \le \frac{(n-3)}{4}.$$

In this case the induced edge label are as follows:

$$f^+(v_{2i-1}v_{2i}) = (2n-2)(2i-1) \; if \; 1 \le i \le \frac{(n-3)}{4};$$

$f^+(v_{2i}v_{2i+1}) = 4(n+1)i \ \ if \ \ 1 \le i \le \frac{(n-3)}{4}$;

$f^+(v_{\frac{n-1+4i}{2}} v_{\frac{n+1+4i}{2}}) = (n-1)(n+1-4i) \ \ if \ \ 1 \le i \le \frac{(n-3)}{4}$;

$f^+(v_{\frac{n+1+4i}{2}} v_{\frac{n+3+4i}{2}}) = (n-3)(n-1-4i) \ \ if \ \ 1 \le i \le \frac{(n-3)}{4}$;

$f^+(v_{\frac{n-1}{2}} v_{\frac{n+1}{2}}) = (n-1)^2$; $f^+(v_{\frac{n+1}{2}} v_{\frac{n+3}{2}}) = (n-2)^2$.

$$\text{Then e-val}_{\max}(W_{n-1}) = \sum_{i=1}^{\left(\frac{n-3}{4}\right)} (2n-2)(2i-1) + \sum_{i=1}^{\left(\frac{n-3}{4}\right)} [4(n+1)\, i\ ] + (n-1)^2$$

$$+ \sum_{i=1}^{\left(\frac{n-3}{4}\right)} (n-1)(n+1-4i) + \sum_{i=1}^{\left(\frac{n-3}{4}\right)} [(n-3)(n-1-4i)] + (n-2)^2$$

$$= \frac{(n^3 - n^2 - 3n + 1)}{2}.$$

Case (ii) $[\, n = 4k+1 \, , \ k \ge 2\,]$

The n vertices of the path P_n are labelled by $f(v_{\frac{n+1}{2}}) = 0$;

$f(v_{2i-1}) = \frac{n+3-4i}{2} \ \ if \ \ 1 \le i \le \frac{(n-1)}{4}$; $f(v_{2i}) = \frac{n-1+4i}{2} \ \ if \ \ 1 \le i \le \frac{(n-1)}{4}$;

$f(v_{\frac{n-1+4i}{2}}) = n-2i \ \ if \ \ 1 \le i \le \frac{(n-1)}{4}$; $f(v_{\frac{n+1+4i}{2}}) = 2i-1 \ \ if \ \ 1 \le i \le \frac{(n-1)}{4}$.

In this case the induced edge label are as follows:

$f^+(v_{2i-1}v_{2i}) = (2n+2)(2i-1) \ if \ \ 1 \le i \le \frac{(n-1)}{4}$;

$f^+(v_{2i}v_{2i+1}) = 4(n-1)i \ \ if \ \ 1 \le i \le \frac{(n-5)}{4}$;

$f^+(v_{\frac{n-1+4i}{2}} v_{\frac{n+1+4i}{2}}) = (n-1)(n+1-4i) \ \ if \ \ 1 \le i \le \frac{(n-1)}{4}$;

$f^+(v_{\frac{n+1+4i}{2}} v_{\frac{n+3+4i}{2}}) = (n-3)(n-1-4i) \ \ if \ \ 1 \le i \le \frac{(n-5)}{4}$;

$f^+(v_{\frac{n-1}{2}} v_{\frac{n+1}{2}}) = (n-1)^2$; $f^+(v_{\frac{n+1}{2}} v_{\frac{n+3}{2}}) = (n-2)^2$.

$$\text{Then e-val}_{\max}(W_{n-1}) = \sum_{i=1}^{\left(\frac{n-1}{4}\right)} (2n+2)(2i-1) + \sum_{i=1}^{\left(\frac{n-5}{4}\right)} [4(n-1)(\, i\,)] + (n-1)^2$$

$$+\sum_{i=1}^{\left(\frac{n-1}{4}\right)}(n-1)(n+1-4i)+\sum_{i=1}^{\left(\frac{n-5}{4}\right)}[(n-3)(n-1-4i)]+(n-2)^2$$

$$=\frac{(n^3-n^2-3n+5)}{2}.$$

Theorem 2.3: Let P_n be a path graph of order n. For every even integer $n \geq 6$,

Then e-val$_{\max}(P_n)=\dfrac{(n^3-n^2-2n+2)}{2}$.

Proof: Let P_n be a path with $n \geq 6$ vertices and $n-1$ edges and where n is an even integer.

Let $V(P_n)=\{v_i : 1 \leq i \leq n\}$. Let $E(P_n)=\{v_i v_{i+1} : 1 \leq i \leq n-1\}$. Then size $m=n-1$.

Let f be a e-labeling of P_n. Let f^+ be the induced edge labeling of f.

Case (i) [$n=4k+2$, $k \geq 1$]

The n vertices of the path P_n are labelled by $f(v_{2i-1})=\dfrac{n+2-4i}{2}$ if $1 \leq i \leq \dfrac{(n+2)}{4}$; $f(v_{2i})=\dfrac{n-2+4i}{2}$ if $1 \leq i \leq \dfrac{(n-2)}{4}$; $f(v_{\frac{n-2+4i}{2}})=n+1-2i$ if $1 \leq i \leq \dfrac{(n+2)}{4}$

$f(v_{\frac{n+4i}{2}})=2i-1$ if $1 \leq i \leq \dfrac{(n-2)}{4}$. In this case the induced edge label are as follows:

$f^+(v_{\frac{n}{2}} v_{\frac{n+2}{2}})=(n-1)^2$; $f^+(v_{2i-1}v_{2i})=2n(2i-1)$ if $1 \leq i \leq \dfrac{(n-2)}{4}$;

$f^+(v_{2i}v_{2i+1})=4(n-2)i$ if $1 \leq i \leq \dfrac{(n-2)}{4}$;

$f^+(v_{\frac{n-2+4i}{2}} v_{\frac{n+4i}{2}})=n(n+2-4i)$ if $1 \leq i \leq \dfrac{(n-2)}{4}$;

$f^+(v_{\frac{n+4i}{2}} v_{\frac{n+2+4i}{2}})=(n-2)(n-4i)$ if $1 \leq i \leq \dfrac{(n-2)}{4}$.

$$\text{Then e-val}_{\max}(W_{n-1})=\sum_{i=1}^{\left(\frac{n-2}{4}\right)}(2n)(2i-1)+\sum_{i=1}^{\left(\frac{n-2}{4}\right)}[4(n-2)i]+(n-1)^2$$

$$+\sum_{i=1}^{\left(\frac{n-2}{4}\right)}4(n-2)i+\sum_{i=1}^{\left(\frac{n-2}{4}\right)}n(n+2-4i)=\frac{(n^3-n^2-2n+2)}{2}.$$

Case (ii) [$n=4k$, $k \geq 2$]

The n vertices of the path P_n are labelled by $f(v_{2i-1})=\dfrac{n-4+4i}{2}$ if $1 \leq i \leq \dfrac{n}{4}$; $f(v_{2i})=\dfrac{n-4i}{2}$ if $1 \leq i \leq \dfrac{n}{4}$; $f(v_{\frac{n-2+4i}{2}})=n-2i+1$ if $1 \leq i \leq \dfrac{n}{4}$;

$f(v_{\frac{n+4i}{2}}) = 2i-1 \;\; if \;\; 1 \le i \le \frac{n}{4}$.

In this case the induced edge label are as follows:

$f^+(v_{\frac{n}{2}} v_{\frac{n+2}{2}}) = (n-1)^2$; $f^+(v_{2i-1} v_{2i}) = 2(n-2)(2i-1) \; if \; 1 \le i \le \frac{n}{4}$;

$f^+(v_{2i} v_{2i+1}) = 4ni \;\; if \;\; 1 \le i \le \frac{(n-4)}{4}$;

$f^+(v_{\frac{n-2+4i}{2}} v_{\frac{n+4i}{2}}) = n(n+2-4i) \;\; if \;\; 1 \le i \le \frac{n}{4}$;

$f^+(v_{\frac{n+4i}{2}} v_{\frac{n+2+4i}{2}}) = (n-2)(n-4i) \;\; if \;\; 1 \le i \le \frac{(n-4)}{4}$.

Then e-val$_{\max}(W_{n-1}) = \sum_{i=1}^{\left(\frac{n}{4}\right)} (2n-2)(2i-1) + \sum_{i=1}^{\left(\frac{n-4}{4}\right)} 4ni + (n-1)^2$

$$+ \sum_{i=1}^{\left(\frac{n}{4}\right)} n(n+2-4i) + \sum_{i=1}^{\left(\frac{n-4}{4}\right)} [(n-2)(n-4i)] = \frac{(n^3 - n^2 - 2n + 2)}{2}.$$

Theorem 2.4: Let C_n be a cycle of order n. Then e-val$_{\min}(C_n) = 2(n-1)^2$.

Proof: Let C_n be a cycle with n vertices and n edges. Let $V(C_n) = \{v_i : 1 \le i \le n\}$.

Let $E(C_n) = \{v_1 v_n ; v_i v_{i+1} : 1 \le i \le n-1\}$ .Then size $m = n$. Let f be a e-labeling of cycle C_n. The n vertices of the cycle C_n are labelled by $f(v_i) = i-1 \;\; if \;\; 1 \le i \le n$.

Let f^+ be the induced edge labeling of f. Then the induced edge label are as follows: $f^+(v_1 v_n) = (n-1)^2$; $f^+(v_i v_{i+1}) = 2i-1 \quad if \;\; 1 \le i \le n-1$.

The minimum label of edges of cycle C_n by f^+ are $\{1,3,5,...,2n-3,(n-1)^2\}$.

Then e-val$_{\min}(C_n) = \sum f^+(uv) = 1+3+5+...+(2n-3)+(n-1)^2 = 2(n-1)^2$.

Theorem 2.5: Let C_n be a cycle of order n. For every even integer $n \ge 4$,

e-val$_{\max}(C_n) = \frac{n^2(n+2)}{2}$.

Proof: Let C_n be cycle with $n \ge 4$ vertices and n edges where n is an even integer.

Let $V(C_n) = \{v_i : 1 \le i \le n\}$. Let $E(C_n) = \{v_1 v_n ; v_i v_{i+1} : 1 \le i \le n-1\}$. Then size $m = n$.

Let f be a e-labeling of C_n. The n vertices of C_n are labelled by $f(v_n) = n$;

$f(v_{2i-1}) = i-1 \;\; if \;\; 1 \le i \le \frac{n}{2}$; $f(v_{2i}) = n-i \;\; if \;\; 1 \le i \le \frac{(n-2)}{2}$. Let f^+ be the induced edge labeling of f. The induced edge labels of C_n by f^+ are as follows:

$f^+(v_1v_n)=n^2$; $f^+(v_{2i-1}v_{2i})=(n^2-1)-2(n-1)i$ if $1\le i\le \frac{(n-2)}{2}$;

$f^+(v_{2i}v_{2i+1})=n^2-2ni$ if $1\le i\le \frac{(n-2)}{2}$; $f^+(v_{n-1}v_n)=\frac{(n+2)(3n-2)}{4}$.

Then e-val$_{\max}(C_n)=\sum_{i=1}^{\left(\frac{n-2}{2}\right)}\left[(n^2-1)-2(n-1)i\right]+\sum_{i=1}^{\left(\frac{n-2}{2}\right)}(n^2-2ni)+\frac{(n+2)(3n-2)}{4}+n^2$

$$=\sum_{i=1}^{\left(\frac{n-2}{2}\right)}(2n^2-1)-\sum_{i=1}^{\left(\frac{n-2}{2}\right)}(4n-2)i+\frac{(7n^2+4n-4)}{4}$$

$$=\frac{(2n^2-1)(n-2)}{2}-\frac{n(n-2)(2n-1)}{4}+\frac{(7n^2+4n-4)}{4}=\frac{n^2(n+2)}{2}.$$

Theorem 2.6: Let C_n be a cycle of order n. For every odd integer $n\ge 3$,

e-val$_{\max}(C_n)=\frac{n(n-1)(n+3)}{2}$.

Proof: Let C_n be cycle with $n\ge 3$ vertices and n edges where n is an odd integer.

Let $V(C_n)=\{v_i : 1\le i\le n\}$. Let $E(C_n)=\{v_1v_n ; v_iv_{i+1} : 1\le i\le n-1\}$. Then size $m=n$.

Let f be a e-labeling of C_n. The n vertices of C_n are labelled by $f(v_n)=n$;

$f(v_{2i-1})=i-1$ if $1\le i\le \frac{(n-1)}{2}$; $f(v_{2i})=n-i$ if $1\le i\le \frac{(n-1)}{2}$. Let f^+ be the induced edge labeling of f. The induced edge labels of C_n by f^+ are as follows:

$f^+(v_1v_n)=n^2$; $f^+(v_{2i-1}v_{2i})=(n^2-1)-2(n-1)i$ if $1\le i\le \frac{(n-1)}{2}$;

$f^+(v_{n-1}v_n)=\frac{(n-1)(3n+1)}{4}$; $f^+(v_{2i}v_{2i+1})=n^2-2ni$ if $1\le i\le \frac{(n-3)}{2}$.

Then e-val$_{\max}(C_n)=\sum_{i=1}^{\left(\frac{n-1}{2}\right)}\left[(n^2-1)-2(n-1)i\right]+\sum_{i=1}^{\left(\frac{n-3}{2}\right)}(n^2-2ni)+\frac{(n-1)(3n+1)}{4}+n^2$

$$=\frac{(n-1)(n^2-1)}{2}-\frac{(n+1)(n^2-1)}{4}+\frac{n^2(n-3)}{2}-\frac{n(n-3)(n-1)}{4}+\frac{(7n^2-2n-1)}{4}$$

$$=\frac{n(n-1)(n+3)}{2}.$$

Example 2.7: The minimum and the maximum e-labeling of cycle C_5 is shown in the Figure-1 and Figure-2 respectively.

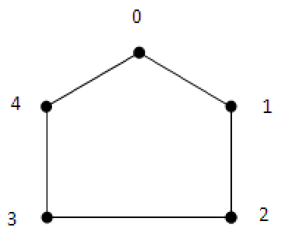

Figure.1

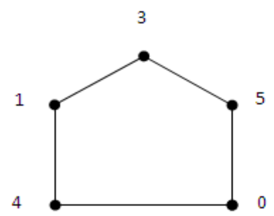

Figure 2

Remark 2.8: From the above example 2.6, observed that

e-val$_{\min}(C_5) = 2(n-1)^2 = 2(16) = 32$ and

e-val$_{\max}(C_5) = \dfrac{n(n-1)(n+3)}{2} = \dfrac{5(5-1)(5+3)}{2} = 80$.

Theorem 2.10: Let $K_{1,n-1}$ be a star graph. For every odd integer $n \geq 3$,

e-val$_{\min}(K_{1,n-1}) = \dfrac{(n-1)(n^2-1)}{4}$.

Proof: Let $K_{1,n-1}$ be a star graph with $n \geq 3$ vertices and $n-1$ edges where n is an odd integer. Let $V(K_{1,n-1}) = \{v_i \ :1 \leq i \leq n\}$. Let $E(K_{1,n-1}) = \{v_1 v_i \ :1 \leq i \leq n-1\}$. Then size $m = n-1$. Let f be a e-labeling of $K_{1,n-1}$. The n vertices of $K_{1,n-1}$ are labelled by

$f(v_1) = \dfrac{(n-1)}{2}$; $f(v_{i+1}) = i-1 \ \text{if} \ 1 \leq i \leq \dfrac{(n-1)}{2}$;

$f(v_{\frac{n+2i+1}{2}}) = \dfrac{n-1+2i}{2} \ \text{if} \ 1 \leq i \leq \dfrac{(n-1)}{2}$.

Let f^+ be the induced edge labeling of f. The induced edge labels of $K_{1,n-1}$ by f^+ are as follows: $f^+(v_1 v_{i+1}) = \dfrac{(n-3+2i)(n+1-2i)}{4} \ \text{if} \ 1 \leq i \leq \dfrac{(n-1)}{2}$;

$f^+(v_1 v_{\frac{n+1+2i}{2}}) = (n-1+i)i \ \text{if} \ 1 \leq i \leq \dfrac{(n-1)}{2}$.

$$\text{Then e-val}_{\min}(K_{1,n-1}) = \sum_{i=1}^{\left(\frac{n-1}{2}\right)} \frac{(n-3+2i)(n+1-2i)}{4} + \sum_{i=1}^{\left(\frac{n-1}{2}\right)} (n-1+i)i$$

$$= \sum_{i=1}^{\left(\frac{n-1}{2}\right)} \frac{(n^2-2n-3)}{4} + (n+1)\sum_{i=1}^{\left(\frac{n-1}{2}\right)} i$$

$$= \frac{(n-1)(n^2-2n-3)}{8} + \frac{(n+1)(n^2-1)}{8} = \frac{(n-1)(n^2-1)}{4}.$$

Theorem 2.11: Let $K_{1,n-1}$ be a star graph. For every even integer $n \geq 4$,

e-val$_{\min}(K_{1,n-1}) = \dfrac{n^2(n-1)}{4}$.

Proof: Let $K_{1,n-1}$ be a star graph with $n \geq 4$ vertices and $n-1$ edges where n is an even integer. Let $V(K_{1,n-1}) = \{v_i \ :1 \leq i \leq n\}$. Let $E(K_{1,n-1}) = \{v_1 v_i \ :1 \leq i \leq n-1\}$. Then size $m = n-1$. Let f be a e-labeling of $K_{1,n-1}$. The n vertices of $K_{1,n-1}$ are labelled by $f(v_1) = \dfrac{n}{2}$; $f(v_{i+1}) = i-1 \ \text{if} \ 1 \leq i \leq \dfrac{n}{2}$;

$f(v_{\frac{n+2i+2}{2}}) = \frac{n+2i}{2}$ *if* $1 \le i \le \frac{(n-2)}{2}$.

Let f^+ be the induced edge labeling of f . The induced edge labels of $K_{1,n-1}$ by f^+

are as follows: $f^+(v_1 v_{i+1}) = \frac{(n-2+2i)(n+2-2i)}{4}$ *if* $1 \le i \le \frac{n}{2}$;

$f^+(v_1 v_{\frac{n+2+2i}{2}}) = (n+i)i$ *if* $1 \le i \le \frac{(n-2)}{2}$.

Then e-val$_{max}(K_{1,n-1}) = \sum_{i=1}^{\left(\frac{n}{2}\right)} \frac{(n-2+2i)(n+2-2i)}{4} + \sum_{i=1}^{\left(\frac{n-2}{2}\right)} (n+i)i$

$$= \frac{n(n^2-4)}{8} + \frac{n(n+2)(n-2)}{8} + \frac{n(4-n^2)}{4} = \frac{n^2(n-1)}{4} .$$

Theorem 2.12: Let $K_{1,n-1}$ be a star graph of order n . Then e-val$_{max}(K_{1,n-1}) = \frac{n(n-1)(4n-5)}{6}$.

Proof: Let $K_{1,n-1}$ be a star graph of order n . Let $V(K_{1,n-1}) = \{v_i : 1 \le i \le n\}$.

Let $E(K_{1,n-1}) = \{v_1 v_i : 1 \le i \le n-1\}$ .Then size $m = n-1$. Let f be a e-labeling of $K_{1,n-1}$.

The n vertices of $K_{1,n-1}$ are labelled by $f(v_1) = n-1$; $f(v_{i+1}) = i-1$ *if* $1 \le i \le (n-2)$.

Let f^+ be the induced edge labeling of f . The induced edge labels of $K_{1,n-1}$ by f^+

are as follows: $f^+(v_1 v_i) = (n-2+i)(n-i)$ *if* $1 \le i \le (n-1)$.

Then e-val$_{max}(K_{1,n-1}) = \sum_{i=1}^{(n-1)} (n-2+i)(n-i) = \sum_{i=1}^{(n-1)} (n^2 - 2n + 2i - i^2)$

$$= n(n-1)(n-2) + n(n-1) + \frac{n(n-1)(2n-1)}{6} = \frac{n(n-1)(4n-5)}{6} .$$

Example 2.13: The minimum and the maximum e-labeling of star $K_{1,5}$ is shown in the Figure-3 and Figure-4 respectively.

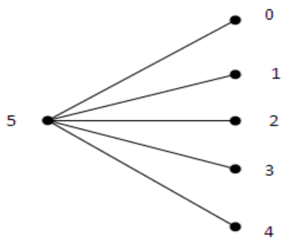

Figure-3

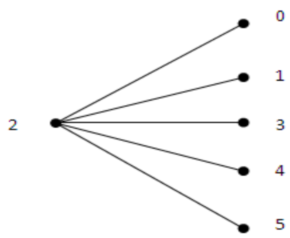

Figure-4

Remark 2.14: From the above example 2.11, observed that

e-val$_{max}(K_{1,5}) = \frac{n(n-1)(4n-5)}{6} = \frac{6(6-1)(4(6)-5)}{6} = 95$

e-val$_{min}$(K$_{1,5}$) $= \dfrac{n^2(n-1)}{4} = \dfrac{(6^2)(6-1)}{4} = 45$.

Theorem 2.15: Let W_{n-1} be a wheel of order $n \geq 4$. Then e-val$_{min}(W_{n-1}) = \dfrac{n(2n^2+9n-23)}{6}$.

Proof: Let W_{n-1} be a wheel of order $n \geq 4$. Let $V(W_{n-1}) = \{v_i \ :1 \leq i \leq n\}$.

Let $E(W_{n-1}) = \{v_1 v_{n-1} \ ; v_i v_{i+1} \ :1 \leq i \leq n-2 \ ; v_i v_n \ :1 \leq i \leq n-1\}$ .Then size $m = 2n-2$.

Define a e-labeling f from $V(W_{n-1})$ to $\{0,1,2,...,2n-2\}$ by $f(v_i) = i \ \ if \ \ 1 \leq i \leq (n-1)$

and $f(v_n) = 0$. Let f^+ be the induced edge labeling of f .

The induced edge labels of W_{n-1} by f^+ are as follows: $f^+(v_1 v_{n-1}) = n(n-2)$;

$f^+(v_i v_{i+1}) = 2i+1 \ \ if \ \ 1 \leq i \leq (n-2)$; $f^+(v_i v_n) = i^2 \ \ if \ \ 1 \leq i \leq (n-1)$.

The minimum edge label are $\{3,5,7,...,2n-3,1^2,2^2,...,(n-1)^2, n(n-2)\}$.

Therefore e-val$_{min}(W_{n-1}) = 3+5+7+...+2n-3+1^2+2^2+...+(n-1)^2+n(n-2)$

$$= \frac{n(n-1)(2n-1)}{6} + 2(n-1)^2 - 2 = \frac{n(2n^2+9n-23)}{6}.$$

Theorem 2.16: Let W_{n-1} be a wheel graph. For every even integer $n \geq 4$,

e-val$_{max}(W_{n-1}) = \dfrac{(65n^3 - 213n^2 + 226n - 72)}{12}$.

Proof: Let W_{n-1} be a wheel of order $n \geq 4$. Let $V(W_{n-1}) = \{ v_i \ : 1 \leq i \leq n \}$.

Let $E(W_{n-1}) = \{v_1 v_{n-1} \ ; v_i v_{i+1} \ :1 \leq i \leq n-2 \ ; v_i v_n \ :1 \leq i \leq n-1\}$ .Then size $m = 2n-2$.

Define a e-labeling f from $V(W_{n-1})$ to $\{0,1,2,...,2n-2\}$ by $f(v_{2i-1}) = i-1 \ \ if \ \ 1 \leq i \leq \dfrac{n}{2}$;

$f(v_{2i}) = 2n-2-i \ \ if \ \ 1 \leq i \leq \dfrac{(n-2)}{2}$ and $f(v_n) = 2n-2$. Let f^+ be the induced edge labeling

of f . The induced edge labels of W_{n-1} by f^+ are as follows: $f^+(v_1 v_{n-1}) = \dfrac{(n-2)^2}{4}$;

$f^+(v_{2i-1} v_{2i}) = (2n-3)(2n-1-2i) \ \ if \ \ 1 \leq i \leq \dfrac{(n-2)}{2}$;

$f^+(v_{2i} v_{2i+1}) = 4(n-1)(n-1-i) \ \ if \ \ 1 \leq i \leq \dfrac{(n-2)}{2}$;

$f^+(v_n v_{2i-1}) = (2n-3+i)(2n-1-i) \ \ if \ \ 1 \leq i \leq \dfrac{n}{2}$;

$f^+(v_n v_{2i}) = (4n-4-i)i \ \ \ if \ \ 1 \leq i \leq \dfrac{(n-2)}{2}$.

Then e-val$_{\max}(W_{n-1}) = \sum_{i=1}^{\left(\frac{n-2}{2}\right)}(2n-3)(2n-1-2i) + \sum_{i=1}^{\left(\frac{n-2}{2}\right)}4(n-1)(n-1-i)$

$$+\sum_{i=1}^{\left(\frac{n}{2}\right)}(2n-3+i)(2n-1-i) + \sum_{i=1}^{\left(\frac{n-2}{2}\right)}(4n-4-i)\,i + \frac{(n-2)^2}{4}$$

$$= \sum_{i=1}^{\left(\frac{n-2}{2}\right)}\left[3(2n-2)^2 - 2(i-1)^2 - 2(2n-2)\right] + (2n-2)^2$$

$$= \frac{(n-2)(12n^2-24n+10)}{2} - \frac{n(n-2)(n-1)}{12} + \frac{n(n-2)(8-4n)}{8} + (4n^2-8n+4)$$

$$= \frac{(65n^3-213n^2-226n-72)}{12}.$$

Theorem 2.17: Let W_{n-1} be a wheel graph. For every odd integer $n \geq 5$,

e-val$_{\max}(W_{n-1}) = \dfrac{(65n^3-201n^2+205n-69)}{12}$.

Proof: : Let W_{n-1} be a wheel of order $n \geq 5$. Let $V(W_{n-1}) = \{ v_i : 1 \leq i \leq n \}$.

Let $E(W_{n-1}) = \{v_1v_{n-1} \; ; v_iv_{i+1} : 1 \leq i \leq n-2 \; ; v_iv_n : 1 \leq i \leq n-1 \}$ .Then size $m = 2n-2$.

Define a e-labeling f from $V(W_{n-1})$ to $\{0,1,2,...,2n-2\}$ by $f(v_{2i-1}) = i-1$ *if* $1 \leq i \leq \frac{(n-1)}{2}$;

$f(v_{2i}) = 2n-2-i$ *if* $1 \leq i \leq \frac{(n-1)}{2}$ and $f(v_n) = 2n-2$. Let f^+ be the induced edge labe-

ling of f. The induced edge labels of W_{n-1} by f^+ are as follows: $f^+(v_1v_{n-1}) = \frac{(3n-3)^2}{4}$;

$f^+(v_{2i-1}v_{2i}) = (2n-3)(2n-1-2i)$ *if* $1 \leq i \leq \frac{(n-1)}{2}$;

$f^+(v_{2i}v_{2i+1}) = 4(n-1)(n-1-i)$ *if* $1 \leq i \leq \frac{(n-3)}{2}$;

$f^+(v_nv_{2i-1}) = (2n-3+i)(2n-1-i)$ *if* $1 \leq i \leq \frac{(n-1)}{2}$;

$f^+(v_nv_{2i}) = (4n-4-i)\,i$ *if* $1 \leq i \leq \frac{(n-1)}{2}$.

Then e-val$_{\max}(W_{n-1}) = \sum_{i=1}^{\left(\frac{n-1}{2}\right)}(2n-3)(2n-1-2i) + \sum_{i=1}^{\left(\frac{n-3}{2}\right)}4(n-1)(n-1-i)$

$$+\sum_{i=1}^{\left(\frac{n-1}{2}\right)}(2n-3+i)(2n-1-i) + \sum_{i=1}^{\left(\frac{n-1}{2}\right)}(4n-4-i)\,i + \frac{(3n-3)^2}{4}$$

$$= \sum_{i=1}^{\left(\frac{n-1}{2}\right)} \left[3(2n-2)^2 - 2\right] + \sum_{i=1}^{\left(\frac{n-1}{2}\right)} (8-4n)i \ - 2 \sum_{i=1}^{\left(\frac{n-1}{2}\right)} i^2 + \frac{(n^2-2n+1)}{4}$$

$$= \frac{(n-1)(12n^2-24n+10)}{2} + \frac{(8-4n)(n-1)(n+1)}{8} - \frac{n(n-1)(n+1)}{12}$$

$$+ \frac{(n^2-2n+1)}{4}$$

$$= \frac{(65n^3-213n^2-226n-72)}{12}.$$

Example 2.18: The minimum and the maximum e-labeling of wheel W_4 are shown in Figure-5 and Figure-6 respectively.

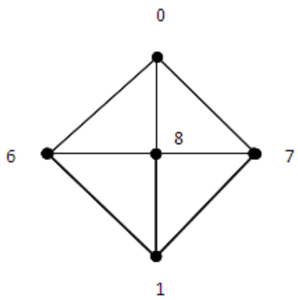

Figure-5

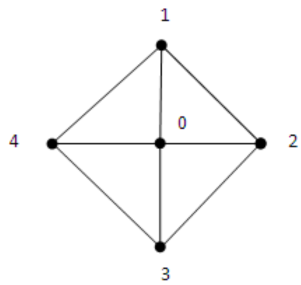

Figure-6

Remark 2.19: From the example 2.14, observed that

$$\text{e-val}_{max}(W_4) = \frac{(65n^3-201n^2+205n-69)}{12} = \frac{(65(5)^3-201(5^2)+205(5)-69)}{12} = 338.$$

$$\text{e-val}_{min}(W_4) = \frac{n(2n^2+9n-23)}{6} = \frac{5(2(5)^2+9(5)-23)}{6} = 60.$$

3. Conclusion

In this paper, we have investigated the maximum and minimum values of e-labeling of certain graphs. We have planned to investigate the maximum and minimum values of e-labeling of some more special graphs in the next paper.

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