International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 5, May 2014, PP 481-488
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Some Determinantal Identities Involving Pell Polynomials

Deepika Jhala

School of studies in Mathematics,
Vikram University, Ujjain, India
jhala.deepika28@gmail.com

G.P.S. Rathore

Department of Mathematical Sciences,
College of Horticulture, Mandsaur, India
gps_rathore20@yahoo.co.in

Kiran Sisodiya

School of studies in Mathematics,
Vikram University, Ujjain, India
sisodiya.kiran4@gmail.com

Abstract: *Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of system of linear equations. There are different perspectives on the study of determinants. In this paper, we obtain determinantal identities involving Pell polynomial and Pell-Lucas polynomial*

Keywords: *Fibonacci number, Lucas number, Fibonacci polynomial, Lucas Polynomial, Pell polynomial, Determinants, Polynomials.*

1. Introduction

In mathematics, polynomials are an important class of simple and smooth functions. Here, simple means they are constructed using only multiplication and addition. Smooth means they are infinitely differentiable, i.e., they have derivatives of all finite orders. Because of their simple structure, polynomials are very easy to evaluate, and are used extensively in numerical analysis for polynomial interpolation or to numerically integrate more complex functions. In linear algebra, the characteristic polynomial of a square matrix encodes several important properties of the matrix.

Fibonacci polynomials are defined by the recurrence relation,

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)\ ;\ n \geq 2\ with\ f_0(x) = 0,\ f_1(x) = 1$$

It is well known that the Fibonacci numbers and polynomials are of great importance in the study of many subjects such as algebra, geometry, combinatorics, approximation theory, graph theory and number theory itself. They occur in a variety of other fields such as finance, art, architecture, music, etc. Fibonacci polynomial has been generalized in a number of ways.

Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of system of linear equations. There are different perspectives on the study of determinants. One may notice several practical and effective instruments for calculating determinants in the nice survey articles [5] and [6]. Much attention has been paid to the evaluation of determinants of matrices, especially when their entries are given recursively [5]. There is a long tradition of using matrices and determinants to study Fibonacci numbers. Bicknell – Johnson and Spears [9] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers. Benjamin, Cameron and Quinn [1], provides combinatorial interpretations for Fibonacci identities using determinants. Koshy [13] explained two chapters on the use of matrices and determinants in Fibonacci numbers.

Spivey [10] describe the sum property for determinants and presented new proofs of identities like the Cassini identity, the d'Ocagne identity and the Catalan identity. Koken and Bozkurt [7] define the Jacobsthal M-matrix and the Jacobsthal Q-matrix similar to the Fibonacci Q-matrix and use these matrix representations to find the Binet-like formula for the Jacobsthal numbers. Macfarlane

[4] use the property for determinants to give new identities involving Fibonacci and related numbers Gupta, Panwar and Sikhwal [12], describes Generalized Fibonacci-Like polynomials and its determinantal identities. Many authors have studied Fibonacci polynomials and Generalized Fibonacci polynomials identities. They applied concept of Matrix and Determinants to establish some identities. In this paper we proved determinantal identities of Pell polynomial and obtain relations of Pell polynomials with other polynomials in determinant form.

2. Generalized Fibonacci Polynomials

Fibonacci polynomial [3] is defined as,

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)\ ;\ n \geq 2\ with\ f_0(x) = 0,\ f_1(x) = 1 \tag{1}$$

Lucas polynomials [3] is defined as,

$$l_n(x) = xl_{n-1}(x) + l_{n-2}(x)\ ;\ n \geq 2\ with\ l_0(x) = 2,\ l_1(x) = x \tag{2}$$

Pell Polynomials [2] is defined as,

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)\ ;\ n \geq 2\ with\ P_0(x) = 0,\ P_1(x) = x \tag{3}$$

Pell-Lucas Polynomials [2] is defined as,

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)\ ;\ n \geq 2\ with\ Q_0(x) = 2,\ Q_1(x) = 2x \tag{4}$$

Chebyshev Polynomials [8] of first kind is defined as,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)\ ;\ n \geq 2\ with\ T_0(x) = 1,\ T_1(x) = x \tag{5}$$

Chebyshev Polynomials [8] of second kind is defined as,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)\ ;\ n \geq 2\ with\ U_0(x) = 1,\ U_1(x) = 2x \tag{6}$$

Vieta-Lucas Polynomials [11] is defined as,

$$\Omega_n(x) = x\Omega_{n-1}(x) - \Omega_{n-2}(x)\ ;\ n \geq 2\ with\ \Omega_0(x) = 2,\ \Omega_1(x) = x \tag{7}$$

3. Determinantal Identities

Before presenting our main theorems we need to introduce some known results and notations we define a family of Pell polynomial as

$$B = \ P_{n+p}(x), P_{n+q}(x), P_{n+q+r}(x), P_{n+s}(x), P_{n+s+r}(x)$$

Where n and p are non-negative integers, q and s are positive integers

with $0 \leq p < q,\ q+1 < s,\ r = 1$

Assume $P_{n+p}(x) = \alpha,\ P_{n+q}(x) = \beta,$ then by (3) $P_{n+q+r}(x) = 2xP_{n+q}(x) + P_{n+p}(x) = \alpha + 2\beta x$ and $P_{n+s}(x) = 2xP_{n+q+r}(x) + P_{n+q}(x)$, $P_{n+s+r}(x) = 2xP_{n+s}(x) + P_{n+q+r}(x)$

Theorem 1: If n and p are non-negative integers, q is positive integer with $0 \leq p < q, r = 1$, prove that

$$\begin{vmatrix} P_{n+p}(x) & P_{n+p}(x) + P_{n+q}(x) & P_{n+p}(x) + P_{n+q}(x) + P_{n+q+r}(x) \\ 2P_{n+p}(x) & 2P_{n+p}(x) + 3P_{n+q}(x) & 2P_{n+p}(x) + 3P_{n+q}(x) + 4P_{n+q+r}(x) \\ 3P_{n+p}(x) & 3P_{n+p}(x) + 6P_{n+q}(x) & 3P_{n+p}(x) + 6P_{n+q}(x) + 12P_{n+q+r}(x) \end{vmatrix} = 3P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x)$$

Proof: Let $\Delta = \begin{vmatrix} P_{n+p}(x) & P_{n+p}(x)+P_{n+q}(x) & P_{n+p}(x)+P_{n+q}(x)+P_{n+q+r}(x) \\ 2P_{n+p}(x) & 2P_{n+p}(x)+3P_{n+q}(x) & 2P_{n+p}(x)+3P_{n+q}(x)+4P_{n+q+r}(x) \\ 3P_{n+p}(x) & 3P_{n+p}(x)+6P_{n+q}(x) & 3P_{n+p}(x)+6P_{n+q}(x)+12P_{n+q+r}(x) \end{vmatrix}$

Assume $P_{n+p}(x)=\alpha,\ P_{n+q}(x)=\beta,$ then by (3) $P_{n+q+r}(x)=\alpha+2\beta x=\gamma$, now

$$\Delta = \begin{vmatrix} \alpha & \alpha+\beta & \alpha+\beta+\gamma \\ 2\alpha & 2\alpha+3\beta & 2\alpha+3\beta+4\gamma \\ 3\alpha & 3\alpha+6\beta & 3\alpha+6\beta+12\gamma \end{vmatrix}$$

Applying $R_2 \to R_2-2R_1,\quad R_3 \to R_3-3R_1$, we have

$$\Delta = \begin{vmatrix} \alpha & \alpha+\beta & \alpha+\beta+\gamma \\ 0 & \beta & \beta+2\gamma \\ 0 & 3\beta & 3\beta+9\gamma \end{vmatrix}$$

Applying $R_3 \to R_3-3R_2$,

$$\Delta = \begin{vmatrix} \alpha & \alpha+\beta & \alpha+\beta+\gamma \\ 0 & \beta & \beta+2\gamma \\ 0 & 0 & 3\gamma \end{vmatrix} = 3\alpha\beta\gamma$$

Put $P_{n+p}(x)=\alpha,\ P_{n+q}(x)=\beta,$ and $P_{n+q+r}(x)=\alpha+2\beta x=\gamma$, we get

$$\begin{vmatrix} P_{n+p}(x) & P_{n+p}(x)+P_{n+q}(x) & P_{n+p}(x)+P_{n+q}(x)+P_{n+q+r}(x) \\ 2P_{n+p}(x) & 2P_{n+p}(x)+3P_{n+q}(x) & 2P_{n+p}(x)+3P_{n+q}(x)+4P_{n+q+r}(x) \\ 3P_{n+p}(x) & 3P_{n+p}(x)+6P_{n+q}(x) & 3P_{n+p}(x)+6P_{n+q}(x)+12P_{n+q+r}(x) \end{vmatrix} = 3P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x).$$

Theorem 2: If *n* and *p* are non-negative integers, *q* is positive integer with $0 \le p < q$, $r=1$ prove that

$$\begin{vmatrix} 0 & P_{n+p}(x)P_{n+q}^2(x) & P_{n+p}(x)P_{n+q+r}^2(x) \\ P_{n+p}^2(x)P_{n+q}(x) & 0 & P_{n+q}(x)P_{n+q+r}^2(x) \\ P_{n+p}^2(x)P_{n+q+r}(x) & P_{n+q+r}(x)P_{n+q}^2(x) & 0 \end{vmatrix} = 2P_{n+p}^3(x)P_{n+q}^3(x)P_{n+q+r}^3(x)$$

Proof: Let $\Delta = \begin{vmatrix} 0 & P_{n+p}(x)P_{n+q}^2(x) & P_{n+p}(x)P_{n+q+r}^2(x) \\ P_{n+p}^2(x)P_{n+q}(x) & 0 & P_{n+q}(x)P_{n+q+r}^2(x) \\ P_{n+p}^2(x)P_{n+q+r}(x) & P_{n+q+r}(x)P_{n+q}^2(x) & 0 \end{vmatrix}$

Assume $P_{n+p}(x)=\alpha,\ P_{n+q}(x)=\beta,$ then by (3) $P_{n+q+r}(x)=\alpha+2\beta x=\gamma$,

$$\Delta = \begin{vmatrix} 0 & \alpha\beta^2 & \alpha\gamma^2 \\ \alpha^2\beta & 0 & \beta\gamma^2 \\ \alpha^2\gamma & \gamma\beta^2 & 0 \end{vmatrix}$$

Taking common $\alpha^2,\ \beta^2,\ \lambda^2$ from $C_1,\ C_2,\ C_3$ respectively, we have

$$\Delta = \alpha^2\beta^2\lambda^2 \begin{vmatrix} 0 & \alpha & \alpha \\ \beta & 0 & \beta \\ \gamma & \gamma & 0 \end{vmatrix} = 2\alpha^3\beta^3\gamma^3$$

Put $P_{n+p}(x) = \alpha,\ P_{n+q}(x) = \beta,$ and $P_{n+q+r}(x) = \gamma$, we get

$$\begin{vmatrix} 0 & P_{n+p}(x)P_{n+q}^2(x) & P_{n+p}(x)P_{n+q+r}^2(x) \\ P_{n+p}^2(x)P_{n+q}(x) & 0 & P_{n+q}(x)P_{n+q+r}^2(x) \\ P_{n+p}^2(x)P_{n+q+r}(x) & P_{n+q+r}(x)P_{n+q}^2(x) & 0 \end{vmatrix} = 2P_{n+p}^3(x)P_{n+q}^3(x)P_{n+q+r}^3(x) .$$

Theorem 3: If n and p are non-negative integers, q is positive integer with $0 \le p < q$, $r = 1$, prove that

$$\begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+q+r}(x) & P_{n+p}(x) & P_{n+q}(x) \\ P_{n+q}(x) & P_{n+q+r}(x) & P_{n+p}(x) \end{vmatrix} = P_{n+p}^3(x) + P_{n+q}^3(x) + P_{n+q+r}^3(x) - 3P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x)$$

Proof: Let $$\Delta = \begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+q+r}(x) & P_{n+p}(x) & P_{n+q}(x) \\ P_{n+q}(x) & P_{n+q+r}(x) & P_{n+p}(x) \end{vmatrix}$$

Assume $P_{n+p}(x) = \alpha, P_{n+q}(x) = \beta,$ then by (3) $P_{n+q+r}(x) = \alpha + 2\beta x = \gamma$, now

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$

$$\Delta = \begin{vmatrix} \alpha+\gamma & \beta+\alpha & \gamma+\beta \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$\Delta = \begin{vmatrix} \gamma-\beta & \beta+\alpha & \gamma+\beta \\ \gamma-\alpha & \alpha & \beta \\ \beta-\gamma & \gamma & \alpha \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$

$$\Delta = \begin{vmatrix} 0 & \alpha+\beta+\gamma & \alpha+\beta+\gamma \\ \gamma-\alpha & \alpha & \beta \\ \beta-\gamma & \gamma & \alpha \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 2\alpha+\beta+2\beta x \\ 2\beta x & \alpha-\beta & \beta \\ \beta-\alpha-2\beta x & 2\beta x & \alpha \end{vmatrix}$$

Expand along first row we get

$$\Delta = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

Put $P_{n+p}(x) = \alpha, P_{n+q}(x) = \beta,$ and $P_{n+q+r}(x) = \gamma$, we

$$\begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+q+r}(x) & P_{n+p}(x) & P_{n+q}(x) \\ P_{n+q}(x) & P_{n+q+r}(x) & P_{n+p}(x) \end{vmatrix} = P_{n+p}^3(x) + P_{n+q}^3(x) + P_{n+q+r}^3(x) - 3P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x)$$

.

Theorem 4: If *n* and *p* are non-negative integers, *q* is positive integer with $0 \le p < q$, $r = 1$ prove that

$$\begin{vmatrix} P_{n+p}(x) & Q_{n+p}(x) & 1 \\ P_{n+q}(x) & Q_{n+q}(x) & 1 \\ P_{n+q+r}(x) & Q_{n+q+r}(x) & 1 \end{vmatrix} = 2x[P_{n+p}(x)Q_{n+q}(x) - P_{n+q}(x)Q_{n+p}(x)]$$

Proof: let $$\Delta = \begin{vmatrix} P_{n+p}(x) & Q_{n+p}(x) & 1 \\ P_{n+q}(x) & Q_{n+q}(x) & 1 \\ P_{n+q+r}(x) & Q_{n+q+r}(x) & 1 \end{vmatrix}$$

Assume $P_{n+p}(x) = \alpha, P_{n+q}(x) = \beta,$ then by (3) $P_{n+q+r}(x) = \alpha + 2\beta x = \gamma$,

and $Q_{n+p}(x) = \eta, Q_{n+q}(x) = \mu,$ then by (4) $Q_{n+q+r}(x) = \eta + 2\mu x = \delta$, now

$$\Delta = \begin{vmatrix} \alpha & \eta & 1 \\ \beta & \mu & 1 \\ \gamma & \delta & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} \alpha - \beta & \eta - \mu & 0 \\ \beta & \mu & 1 \\ \gamma & \delta & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} \alpha - \beta & \eta - \mu & 0 \\ \beta - \gamma & \mu - \delta & 0 \\ \gamma & \delta & 1 \end{vmatrix} = 2x(\beta\eta - \alpha\mu)$$

Put $P_{n+p}(x) = \alpha, P_{n+q}(x) = \beta,$ $P_{n+q+r}(x) = \alpha + 2\beta x = \gamma$ and

$Q_{n+p}(x) = \eta, Q_{n+q}(x) = \mu,$ $Q_{n+q+r}(x) = \eta + 2\mu x = \delta$, we get

$$\begin{vmatrix} P_{n+p}(x) & Q_{n+p}(x) & 1 \\ P_{n+q}(x) & Q_{n+q}(x) & 1 \\ P_{n+q+r}(x) & Q_{n+q+r}(x) & 1 \end{vmatrix} = 2x[P_{n+p}(x)Q_{n+q}(x) - P_{n+q}(x)Q_{n+p}(x)].$$

Theorem 5: If n and p are non-negative integers, q is positive integer with $0 \le p < q$, $r = 1$, prove that

$$\begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+q}(x) & P_{n+q+r}(x) & P_{n+s}(x) \\ P_{n+q+r}(x) & P_{n+s}(x) & P_{n+s+r}(x) \end{vmatrix} = 0$$

Proof: Let $$\Delta = \begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+q}(x) & P_{n+q+r}(x) & P_{n+s}(x) \\ P_{n+q+r}(x) & P_{n+s}(x) & P_{n+s+r}(x) \end{vmatrix}$$

Assume that

$$P_{n+p}(x) = \alpha,\ P_{n+q}(x) = \beta, P_{n+q+r}(x) = \alpha + 2\beta x = \gamma, P_{n+s}(x) = 2x\gamma + \beta = \wp \ and\ P_{n+s+r}(x) = 2x\wp + \gamma = \kappa$$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \wp \\ \gamma & \wp & \kappa \end{vmatrix}$$

$Applying\, C_1 \to C_1 + 2xC_2$

$$\Delta = \begin{vmatrix} \gamma & \beta & \gamma \\ \wp & \gamma & \wp \\ \kappa & \wp & \kappa \end{vmatrix} = 0 \qquad (two\ column\ are\ identical\,)$$

$$\begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+q}(x) & P_{n+q+r}(x) & P_{n+s}(x) \\ P_{n+q+r}(x) & P_{n+s}(x) & P_{n+s+r}(x) \end{vmatrix} = 0$$

Theorem 6: If n and p are non-negative integers, q is positive integer with $0 \le p < q$, $r = 1$, prove that

$$\begin{vmatrix} P_{n+p}(x) + P_{n+q}(x)\ ^{2} & P_{n+p}(x)P_{n+q+r}(x) & P_{n+q}(x)P_{n+q+r}(x) \\ P_{n+p}(x)P_{n+q+r}(x) & P_{n+q}(x) + P_{n+q+r}(x)\ ^{2} & P_{n+p}(x)P_{n+q}(x) \\ P_{n+q}(x)P_{n+q+r}(x) & P_{n+p}(x)P_{n+q}(x) & P_{n+q+r}(x) + P_{n+p}(x)\ ^{2} \end{vmatrix} =$$

$$2P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x)\ \ P_{n+p}(x) + P_{n+q}(x) + P_{n+q+r}(x)\ ^{3}$$

Proof: Let $$\Delta = \begin{vmatrix} P_{n+p}(x) + P_{n+q}(x)\ ^{2} & P_{n+p}(x)P_{n+q+r}(x) & P_{n+q}(x)P_{n+q+r}(x) \\ P_{n+p}(x)P_{n+q+r}(x) & P_{n+q}(x) + P_{n+q+r}(x)\ ^{2} & P_{n+p}(x)P_{n+q}(x) \\ P_{n+q}(x)P_{n+q+r}(x) & P_{n+p}(x)P_{n+q}(x) & P_{n+q+r}(x) + P_{n+p}(x)\ ^{2} \end{vmatrix}$$

Assume $P_{n+p}(x) = \alpha, P_{n+q}(x) = \beta,$ then by (3) $P_{n+q+r}(x) = 2xP_{n+q}(x) + P_{n+p}(x) = \gamma$

$$\Delta = \begin{vmatrix} (\alpha+\beta)^2 & \alpha\gamma & \beta\gamma \\ \alpha\gamma & (\beta+\gamma)^2 & \alpha\beta \\ \beta\gamma & \alpha\beta & (\gamma+\alpha)^2 \end{vmatrix}$$

Expanding along first row, we obtained $\Delta = 2\alpha\beta\gamma(\alpha+\beta+\gamma)^3$

$$\begin{vmatrix} (P_{n+p}(x)+P_{n+q}(x))^2 & P_{n+p}(x)P_{n+q+r}(x) & P_{n+q}(x)P_{n+q+r}(x) \\ P_{n+p}(x)P_{n+q+r}(x) & (P_{n+q}(x)+P_{n+q+r}(x))^2 & P_{n+p}(x)P_{n+q}(x) \\ P_{n+q}(x)P_{n+q+r}(x) & P_{n+p}(x)P_{n+q}(x) & (P_{n+q+r}(x)+P_{n+p}(x))^2 \end{vmatrix} =$$

$$2P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x)\,(P_{n+p}(x)+P_{n+q}(x)+P_{n+q+r}(x))^3.$$

Theorem 7: If n and p are non-negative integers, q is positive integer with $0 \le p < q$, $r = 1$, prove that

$$\begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+p}(x)-P_{n+q}(x) & P_{n+q}(x)-P_{n+q+r}(x) & P_{n+q+r}(x)-P_{n+p}(x) \\ P_{n+q}(x)+P_{n+q+r}(x) & P_{n+p}(x)+P_{n+q+r}(x) & P_{n+p}(x)+P_{n+q}(x) \end{vmatrix} =$$

$$P^3_{n+p}(x)+P^3_{n+q}(x)+P^3_{n+q+r}(x)-3P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x)$$

Proof: Let $$\Delta = \begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+p}(x)-P_{n+q}(x) & P_{n+q}(x)-P_{n+q+r}(x) & P_{n+q+r}(x)-P_{n+p}(x) \\ P_{n+q}(x)+P_{n+q+r}(x) & P_{n+p}(x)+P_{n+q+r}(x) & P_{n+p}(x)+P_{n+q}(x) \end{vmatrix}$$

Assume $P_{n+p}(x)=\alpha, P_{n+q}(x)=\beta,$ then by (3) $P_{n+q+r}(x) = 2xP_{n+q}(x)+P_{n+p}(x)=\gamma$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha-\beta & \beta-\gamma & \gamma-\alpha \\ \beta+\gamma & \alpha+\gamma & \alpha+\beta \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} \beta & \gamma & \alpha \\ \alpha-\beta & \beta-\gamma & \gamma-\alpha \\ \beta+\gamma & \alpha+\gamma & \alpha+\beta \end{vmatrix}$$

$$\Delta = \alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma$$

Put $P_{n+p}(x)=\alpha, P_{n+q}(x)=\beta,$ and $P_{n+q+r}(x)=\gamma$, we get

$$\begin{vmatrix} P_{n+p}(x) & P_{n+q}(x) & P_{n+q+r}(x) \\ P_{n+p}(x)-P_{n+q}(x) & P_{n+q}(x)-P_{n+q+r}(x) & P_{n+q+r}(x)-P_{n+p}(x) \\ P_{n+q}(x)+P_{n+q+r}(x) & P_{n+p}(x)+P_{n+q+r}(x) & P_{n+p}(x)+P_{n+q}(x) \end{vmatrix} =$$

$$P^3_{n+p}(x)+P^3_{n+q}(x)+P^3_{n+q+r}(x)-3P_{n+p}(x)P_{n+q}(x)P_{n+q+r}(x).$$

4. Conclusion

This paper describes developed determinant identities of Pell polynomials and derived relational identities of Pell polynomials with others polynomials. Also extended the results in higher order determinants. These identities can be used to develop new identities of polynomials like Fibonacci polynomials Jacobthal polynomial and other Fibonacci-Like polynomial.

Acknowledgement

Guidance during preparation of this research paper fills me with a sense of gratitude. Firstly, I express my gratitude to my supervisor Dr. G.P.S.Rathore and co supervisor Dr. Bijendra Singh for their generous support.

References

[1] A. Benjamin, N. Cameron and J. Quinn, Fibonacci Determinants- A Combinatorial Approach, Fibonacci Quarterly, 45(1), 39-55, (2007).

[2] A.F.Horadam and Bro J. M. Mahon, Pell and Pell-Lucas Polynomials, Fibonacci Quarterly, 23(1), 17-20, (1985).

[3] A. Lupas, A Guide of Fibonacci and Lucas Polynomial, Octagon Mathematics Magazine, 7(1), (1999).

[4] A. J. Macfarlane, Use of Determinants to Present Identities Involving Fibonacci and Related Numbers, Fibonacci Quarterly, 48(1), 68-7648, (2010).

[5] C. Krattenthaler , Advanced determinant calculus, A Complement, Liner Algebra Appl., 41, 168-166, (2005).

[6] C. Krattenthaler, Advanced determinant calculus, Seminaire Lotharingien Combin, Article, b42q, 67, (1999).

[7] F. Koken and D. Bozkur, On the Jacobsthaal Numbers by Matrix Methods, Int. J. Contemp. Math. Sciences, 3(13), 605-614, (2008).

[8] G. Udrea, Chebshev Polynomials and Some Methods of Approximation, Portugaliae Mathematica, 55(3), 261-269, 1998.

[9] M. Bicknell-Johnson and C.P. Spears, Classes Of Identities For the Generalized Fibonacci number $G_n = G_{n-1} + G_{n-2}\ ;\ n \geq 2$ from Matrices with Constant valued Determinants, Fibonacci Quarterly, 34(2), 121-128, (1996).

[10] M. Z. Spivey, Fibonacci Identities via the Determinant sum property, College Mathematics Journal, 37(4), 286-289, (2006).

[11] R. Witula and D. Slota, Conjugate Sequences in a Fibonacci-Lucas Sense and Some Identities for Sums of Powers of their elements, Integers: Electronic Journal of Combinatorial Number Theory, 7, 1-26, 2007.

[12] V. K. Gupta, Y. K. Panwar and O.P. Sikhwal, Generalized Fibonacci-Like Polynomial and its Determinantal Identities, Int. J. Contemp. Math. Sciences, 7(29), 1415-1420, 2012.

[13] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, (2001).