*International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*
*Volume 2, Issue 4, April 2014, PP 366-377*
*ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)*
*www.arcjournals.org*

# Number of Cycles of Length Four in Sum Graphs $G_n$ and Integral Sum Graphs $G_{m,n}$

**V. Vilfred**
Department of Mathematics
St.Jude's College, Thoothoor
Kanyakumari District,
Tamil Nadu, India.
*vilfredkamal@gmail.com*

**K. Rubin Mary**
Department of Mathematics
St.Jude's College, Thoothoor
Kanyakumari District,
Tamil Nadu, India.
*rubyjudes@yahoo.com*

**Abstract:** *A sum graph is a graph for which there is a labeling of its vertices with positive integers so that two vertices are adjacent if and only if the sum of their labels is the label of another vertex. Integral sum graphs are defined similarly, except that the labels may be any integers. These concepts were first introduced by Harary, who provided examples of such graphs of all orders. The family of integral sum graphs $G_{-n,n}$ was extended to $G_{-m,n}$ by Vilfred who calculated number of triangles in $G_k$, $G_k^c$,$G_{-m,n}$ and $G^c_{-m,n}$, $k\in N$ and $m,n\in N_0$. In this paper, we calculate number of cycles of length four, at first, in graphs $G_k$ and $G_k^c$ and then using these we obtain that of $G_{-m,n}$ and $G^c_{-m,n}$, $k\in N$ and $m,n\in N_0$. Also, we prove that for $n\in N$,$G_{0,n}\cong G_{n+2}\backslash\{u_{n+2}\}$ and $G_{-1,n}\cong G_{n+4}\backslash\{u_{n+3},u_{n+4}\}$ with-out vertex labels where $u_j$ is the vertex with integral sum labeling j in $G_m$ and anti-integral sum labeling j in $G_m^c$, m = n+2 or m = n+4 and $1 \le j \le m$ and obtain a few properties of natural numbers.*

AMS Subject Classification: **05C75, 05C78**.

**Keywords:** *Integral sum graph, anti-integral sum graph, split graph, clique, stable set or independent set, $G^+(S)$, $G_{\Delta n}$, $G_n$, $G_{m,n}$, $/H/_G$.*

**Research supported in part by DST, Govt. of India, under grant SR/S4/MS: 679/10 (DST-SERB) and Lerroy Wilson Foundation, Nagercoil, India (Face Book: Lerroy Wilson Foundation, India).*

## 1 INTRODUCTION

Harary introduced the concept of sum graph in **[1]**. A graph *G* = (*V*,*E*) is a *sum graph* or *N-graph* if the vertices of *G* can be labeled with distinct positive integers so that *e* = *uv* is an edge of *G* if and only if the sum of the labels on vertices *u* and *v* is also a label in *G*. Harary **[2]** extended the sum graph concept to allow any integers to be used as labels. He provided examples of graphs of this type. To distinguish between the two types, we refer to sum graphs that use only positive integers as *N-sum graphs* and those that use any integers as *Z-sum graphs* **[3]**. For any non-empty set of integers *S*, we let $G^+(S)$ denote the integral sum graph on the set *S*. For integers *r* and *s* with *r*<*s* we also let [*r*, *s*] denote the set of integers {*r*,*r*+1,...,*s*}. Harary's examples of *N-sum* graphs are thus $G^+([1,n]) = G_n$ and his *Z-sum* graphs are $G^+([-r,r]) = G_{-r,r}$ for $r\in N$. (Note that his notation is modified and we write $G_{-r,r}$ for what he called $G_{r,r}$. See **[3]**). Beineke, Chen, Harary, Kala, Mary Florida, Nicholas, Rubin Mary, Suryakala and Vilfred **[1]-[14]** studied general properties of sum and integral sum graphs. The extension of Harary graphs to all intervals of integers was introduced by Vilfred and Mary Florida in **[8]**: for any integers *r* and *s* with *r*<*s*, let $G_{r,s} = G^+([r,s])$. We denote the sum graph $G^+([1,n])$ by $G_n^+$ when it is labeled and by $G_n$ when it is unlabeled and [*k*] in $G^+(S)$ denotes the set of all edges of $G^+(S)$ whose edge sum value is *k*, $k\in S$**[9]**. See Figures 1 and 2.

Vilfred **[7]** introduced the concepts of anti-sum and anti-integral sum labeling and calculated the number of triangles in $G_k$, $G_k^c$,$G_{-m,n}$ and $G^c_{-m,n}$,$k\in N$ and $m,n\in N_0$ [6]. In this paper, we prove that for $n\in N$,$G_{0,n}\cong G_{n+2}\backslash\{u_{n+2}\}$ and $G_{-1,n}\cong G_{n+4}\backslash\{u_{n+3},u_{n+4}\}$ with-out vertex labels where $u_j$ is the vertex with integral sum labeling *j* in $G_m$, *m* = *n*+2 or *m* = *n*+4 and $1 \le j \le m$; $|C_4|_{G_{2n+2}}=$

$\frac{(n-1)n(n+1)(7n-10)}{24} = |C_4|_{G^c_{2n+1}}$; $|C_4|_{G_{2n+3}} = \frac{(n-1)n(n+1)(7n+6)}{24} = |C_4|_{G^c_{2n+2}}$; $|C_4|_{G_{0,n}} = |C_4|_{G_{n+2}}$; $|C_4|_{G_{-1,n}} = |C_4|_{G_{n+4}}$; $|C_4|_{G_{-m,n}} = |C_4|_{G_m} + |C_4|_{G_n} + 3(|C_3|_{G_m} + |C_3|_{G_n}) + 2(n.|E(G_m)| + m.|E(G_n)|) + m\mathrm{C}_2.n\mathrm{C}_2 + n.m\mathrm{C}_2 + m.n\mathrm{C}_2$ + (number of $P_3$s in $-G_m$, each $P_3 = uvw$ with $uw \notin E(-G_m)$) + (number of $P_3$s in $G_n$, each $P_3 = uvw$ with $uw \notin E(G_n)$); $|C_4|_{G^c_{-m,n}} = |C_4|_{G^c_m} + |C_4|_{G^c_n}$; $|C_4|_{G_{-2m,2n}} = \frac{(m-1)m(7m^2+m-18)}{24} + \frac{(n-1)n(7n^2+n-18)}{24} + mn(4mn+6(m+n) - 11)$; $|C_4|_{G_{-2m,2n+1}} = \frac{(m-1)m(7m^2+m-18)}{24} + \frac{(n-1)n(7n^2+17n-2)}{24} + m(4m-3)(2n+1) + mn(4mn+2m+6n+1)$; $|C_4|_{G_{-(2m+1),2n}} = \frac{(m-1)m(7m^2+17m-2)}{24} + \frac{(n-1)n(7n^2+n-18)}{24} + mn(4mn+6m+2n+1) + (2m+1)n(4n-3)$ and $|C_4|_{G_{-(2m+1),2n+1}} = \frac{(m-1)m(7m^2+17m-2)}{24} + \frac{(n-1)n(7n^2+17n-2)}{24} + 2(m^2+n^2) + (mn+m+n)(2m+1)(2n+1) + 4mn(m+n)$ where $|H|_G$ denotes number of distinct sub-graphs, each isomorphic to $H$, in graph $G$, $2 \le m,n$. We obtain the following properties of natural numbers: for $2 \le n$ and $n \in N$, 6 divides $n(n+1)(7n-4)$, $n(n+1)(7n+8)$ and $n(7n^2+18n+5)$ and 24 divides $n(n+1)(n+2)(7n-3)$, $n(n+1)(n+2)(7n+1)$, $n(n+1)(n+2)(7n+13)$, $n(n+1)(7n^2+15n-10)$ and $n(n+1)(7n^2+31n+22)$ **[14]**.

All graphs in this paper are simple graphs. For all basic notation and definitions in graph theory, we follow **[15]** and for sum and integral sum graphs, we refer to **[3]**, **[16]**. Now, we consider a few definitions and properties of sum and integral sum graphs.

A graph$G$is an *anti-sum graph* or*anti-N-sum graph* if thevertices of$G$can be labeled with distinct positive integers so that $e = uv$ is an edge of$G$if and only if the sum of the labels on vertices $u$ and $v$ is not a vertex label in $G$ **[7]**. An *anti-integral sum graph* or *anti-Z-sum graph* is also defined just as anti-sum graph, the difference being that the labels may be any distinct integers. Clearly, $f$ is an integral sum labeling of graph $G$ if and only if $f$ is an anti-integral sum labeling of $G^c$.

A graph $G$ is a *split graph* if its vertices can be partitioned into a clique and a stable set. A *clique* in a graph is a set of pair-wise adjacent vertices and an *independent set* or *stable set*in a graph is a set of pair-wise non-adjacent vertices **[17]**. $G_n$and$G^c_n$are split graphs. Clearly, $[1,m]$, $[1,m+1]$, $[m+1,2m]$, $[m+2,2m+1]$ are cliques and $[m+1,2m]$, $[m+2,2m+1]$, $[1,m]$, $[1,m+1]$ are stable sets in $G_{2m}$, $G_{2m+1}$, $G^c_{2m}$,$G^c_{2m+1}$, respectively.

Two vertices with label $j$ and $k$, in a sum graph $G^+(S)$ with $n$ as its maximum vertex label, are called *supplementary vertices* if $j+k = n+1$ and the corresponding labels are called *supplementary labels*, $1 \le j,k \le n$, $j \ne k$ and $n \ge 2$ **[3]**. In $G_n$, $|E(G_n)| = ½(n(n-1)/2 - \lfloor n/2 \rfloor)$, $d(v_j) = n-1-j$ if $1 \le j \le \lfloor \frac{n+1}{2} \rfloor$ and $d(v_j) = n-j$ if $\lfloor \frac{n+1}{2} \rfloor + 1 \le j \le n$ where $\lfloor x \rfloor$ is the floor of $x$, $V(G_n) = \{v_1, v_2, \ldots, v_n\}$ and $j$ is the vertex sum label of $v_j$in $G_n$, $1 \le j \le n$ and $2 \le n$.

**Theorem 1.1 [8]** If *-r,s∈N with r < 0 <s, then*$G_{r,s} = K_1 + (G_{-r} + G_s)$. □

**Theorem 1.2 [12]** *Every integral sum graph G of order n, except* $K_3$, *has at the most two vertices of degree n*-1. □

**Theorem 1.3 [12]** *For every n*≥ 4, *there is an integral sum graph of order n with exactly two vertices of degree n*-1. *This graph is unique up to isomorphism and is denoted by* $G_{\Delta n}$. □

**Theorem 1.4 [8]** *For*$m,n \ge 2$, $G_{0,n}$*and*$G_{-m,n}$*contain exactly one vertex of degree n and m+n, respectively. For* $2 \le n$, $G_{-1,n}$*has exactly two vertices of degree n*+1. $G_{-1,1}$*is the only integral sum graph G having more than two vertices of degree* 2. □

**Theorem 1.5 [8]** *For* $3 \le m+n$, $|E(G_{-m,n})| = ¼(m^2+n^2+3(m+n)+4mn) - ½(\lfloor m/2 \rfloor + \lfloor n/2 \rfloor)$ *where* $\lfloor x \rfloor$*denotes the floor of x,* $m,n \in N_0$. *In particular,* $|E(G_{0,n})| = \frac{n(n+3)}{4} - ½(\lfloor n/2 \rfloor)$, $|E(G_{-n,n})| = 3n(n+1)/2 - \lfloor n/2 \rfloor$*and* $|E(G_{-(n-1),n})| = n(3n-1)/2$, $n \in N$. □

**Theorem 1.6 [3]** *Let k and n be such that* $2 \le 2k < n$.*If k pairs of supplementary vertices are removed from* (i) *Harary graph* $G_n$, *then the result is isomorphic to* $G_{n-2k}$ *without the vertex labels and* (ii) the graph $G^c_n$, *then the result is isomorphic to* $G^c_{n-2k}$ *without the vertex labels.* □

**Theorem1.7 [3]** *For* $n \geq 3$, *the underlying graphs of* $G_{0,n} \backslash \{0,n\}$ *and* $G_{0,n-2}$ *are isomorphic and for* $n \geq 2r+3$ *and* $r \in N$, *the underlying graphs of* $G_{0,n} \backslash (\{0, n, n-1, n-2, \ldots, n-2r+1, n-2r\} \cup ([n] \cup [n-1] \cup \ldots \cup [n-2r+1]))$ *and* $G_{0,n-2r-2}$ *are isomorphic.* □

**Theorem 1.8 [6]** *For* $3 \leq n$, $|C_3|_{G_n} = |C_3|_{G_{n-2}} + |E(G_{n-2})|$ and $|C_3|_{G_n^c} = |C_3|_{G_{n-2}^c} + |E(G_{n-2}^c)|$. □

**Corollary1.9 [6]** *For* $n \in N$, $|C_3|_{G_{2n+2}} = \frac{(n-1)n(n+1)}{3} = |C_3|_{G_{2n+1}^c}$ *and* $|C_3|_{G_{2n+3}} = \frac{n(n+1)(2n+1)}{6} = |C_3|_{G_{2n+2}^c}$. □

**Theorem 1.10 [6]** *For* $m,n \in N_0$, $|C_3|_{G_{-m,n}} = |C_3|_{G_m} + |C_3|_{G_n} + (n+1).|E(G_m)| + (m+1).|E(G_n)| + mn$ *and* $|C_3|_{G_{-m,n}^c} = |C_3|_{G_m^c} + |C_3|_{G_n^c}$. □

**Corollary 1.11 [6]** *For* $m,n \in N$,

(i) $|C_3|_{G_{-2m,2n}} = \frac{1}{3}(m+n)(m^2+5mn+n^2-1)$;

(ii) $|C_3|_{G_{-2m,2n+1}} = \frac{1}{6}(2(m^3+n^3)+12mn(m+n)+3(2m^2+n^2+4mn)+4m+n)$;

(iii) $|C_3|_{G_{-(2m+1),2n}} = \frac{1}{6}(2(m^3+n^3)+12mn(m+n)+3(m^2+2n^2+4mn)+m+4n)$;

(iv) $|C_3|_{G_{-(2m+1),2n+1}} = \frac{1}{6}(m+n)(2(m+n)^2+9(m+n)+6mn+13)+mn+1$;

(v) $|C_3|_{G_{-2m,2n}^c} = \frac{(m-1)m(2m-1)}{6} + \frac{(n-1)n(2n-1)}{6}$;

(vi) $|C_3|_{G_{-2m,2n+1}^c} = \frac{(m-1)m(2m-1)}{6} + \frac{(n-1)n(n+1)}{3}$;

(vii) $|C_3|_{G_{-(2m+1),2n}^c} = \frac{(m-1)m(m+1)}{3} + \frac{(n-1)n(2n-1)}{6}$ and

(viii) $|C_3|_{G_{-(2m+1),2n+1}^c} = \frac{(m-1)m(m+1)}{3} + \frac{(n-1)n(n+1)}{3}$. □

## 2 COUNTING NUMBER OF $C_4$S IN $G_n$ AND $G_{-m,n}$

We count the number of cycles of length four in $G_{2k}$, $G_{2k+1}$, $G_{2k}^c$ and $G_{2k+1}^c$ and using these, we obtain the number of cycles of length four in $G_{-m,n}$ and $G_{-m,n}^c$, $2 \leq k$ and $m,n \in N_0$. We have $G_{-m,n} = K_1 + ((-G_m) + G_n)$, $G_{-m,n}^c = K_1(0) \cup (-G_m^c) \cup G_n^c$, $|E(G_n)| = ½(nC_2 - \lfloor \frac{n}{2} \rfloor)$, $|E(G_n^c)| = ½(nC_2 + \lfloor \frac{n}{2} \rfloor)$, $|E(G_{2n})| = n^2 - n = |E(G_{2n-1}^c)|$ and $|E(G_{2n})| = n^2 = |E(G_{2n}^c)|$ where $\lfloor x \rfloor$ denotes the floor of $x$, $m,n \in N_0$ **[8]**.

**Theorem 2.1** *For* $2 \leq n$, $|C_4|_{G_{2n+2}} = |C_4|_{G_{2n}} + \frac{(n-1)n(7n-11)}{6} = \frac{(n-1)n(n+1)(7n-10)}{24}$ *and* $|C_4|_{G_{2n+2}^c} = |C_4|_{G_{2n}^c} + \frac{(n-1)n(7n+1)}{6} = \frac{(n-1)n(n+1)(7n+6)}{24}$.

Proof:Let $V(G_{2n+2}) = \{u_1,u_2,\ldots,u_{2n+2}\} = V(G_{2n+2}^c)$ where $u_j$ is the vertex with sum labeling $j$ in $G_{2n+2}$ and anti-sum labeling $j$ in $G_{2n+2}^c$, $1 \leq j \leq 2n+2$ and $n \in N$. At first, let us to prove the result for $G_{2n+2}$, $n \in N$. $\{u_1,u_2,\ldots,u_{n+1}\}$ is a clique and $\{u_{n+2},u_{n+3},\ldots,u_{2n+2}\}$ is a stable set to $G_{2n+2}$. Using Theorem 1.6, graph $G_{2n+2} \backslash \{u_1,u_{2n+2}\}$ is isomorphic to $G_{2n}$, without the vertex labels. In $G_{2n+2}$, $u_1$ is adjacent to $u_2,u_3,\ldots,u_{2n+1}$; $u_{2n+2}$ is an isolated vertex and $u_{2n+1}$ is a pendant vertex. Therefore, $|C_4|_{G_{2n+2}} = |C_4|_{G_{2n}}$ + number of cycles of length four, each with $u_1$ as a vertex in $G_{2n+2}$. Also, none of $u_{2n+1}$ and $u_{2n+2}$ is a vertex of any cycle of length 4 in $G_{2n+2}$.

Let $(u_1u_iu_ju_k)$ be any cycle of length 4 (with $u_1$ as a vertex) in $G_{2n+2}$, $1 < i,j,k < 2n+1$ and $i,j,k$ are all different. Under the above conditions, the following three types of $C_4$s arise in $G_{2n+2}$. Type-1: $u_i,u_j,u_k \in \{u_2,u_3,\ldots,u_{n+1}\}$, Type-2: $u_i,u_j \in \{u_2,u_3,\ldots,u_{n+1}\}$ and $u_k \in \{u_{n+2},u_{n+3},\ldots,u_{2n}\}$ and Type-3: $u_i \in \{u_2,u_3,\ldots,u_{n+1}\}$ and $u_j,u_k \in \{u_{n+2},u_{n+3},\ldots,u_{2n}\}$. Now, let us calculate number of $C_4$s in $G_{2n+2}$ under each type.

**Number of $C_4$s of Type-1:** Here , $u_i,u_j,u_k \in \{u_2,u_3,\ldots,u_{n+1}\}$ in $G_{2n+2}$. Number of ways of selecting 3 vertices $u_i,u_j,u_k$ out of $u_2,u_3,\ldots,u_{n+1}$ is $n\text{C}_3$. There are 3 different $C_4$s with $u_1$, $u_i$, $u_j$, $u_k$ as vertices under type-1, namely, $(u_1u_iu_ju_k)$, $(u_1u_iu_ku_j)$ and $(u_1u_ju_iu_k)$. Therefore, total number of $C_4$s of type-1 in $G_{2n+2}$ = $3.n\text{C}_3$= $\frac{n(n-1)(n-2)}{2}$.

**Number of $C_4$s of Type-2:** Here, $u_i,u_j \in \{u_2,u_3,\ldots,u_{n+1}\}$ and $u_k \in \{u_{n+2},u_{n+3},\ldots,u_{2n}\}$. Consider all possible cycles, each of length 4 and with vertices $u_1$, $u_i$, $u_j$ and $u_k$in $G_{2n+2}$.

When $k = 2n$, $u_k = u_{2n}$ is adjacent to $u_1$ and $u_2$ only. And under this case, $u_2 = u_i$ or $u_2 = u_j$. W.l.g., assume $u_2 = u_i$. This implies, $2 = i < 3 \leq j \leq n+1$. And any $C_4$ under this case is of the form $(u_1u_ku_iu_j) = (u_1u_{2n}u_2u_j)$, $u_j \in \{u_3,u_4,\ldots,u_{n+1}\}$ and number of such $C_4$s is $|\{u_3,u_4,\ldots,u_{n+1}\}| = n-1$.

When $k = 2n-1$, $u_k = u_{2n-1}$ is adjacent to $u_1$, $u_2$ and $u_3$ only and thereby $d(u_k) = 3 = 2n+2-(2n-1)$. And any $C_4$ of type-2 is of the form $(u_1u_{2n-1}u_2u_x)$ or $(u_1u_{2n-1}u_3u_y)$, $u_x \in \{u_3,u_4,\ldots,u_{n+1}\}$ and $u_y \in \{u_2,u_4,u_5,\ldots,u_{n+1}\}$. Number of such $C_4$s is $2(n-1)$.

When $k = 2n-2$, $u_k = u_{2n-2}$ is adjacent to $u_1$ ,$u_2$ , $u_3$ and $u_4$ only and thereby $d(u_k) = 4 = 2n+2-(2n-2)$. Therefore, number of such $C_4$s is $(4-1)(n-1) = 3(n-1)$.

In general, when $k = 2n+2-x$ and $2 \leq x \leq n$, $u_k = u_{2n+2-x}$is adjacent to $u_1$ ,$u_2$ , . . . , $u_x$ only and thereby $d(u_k) = d(u_{2n+2-x}) = x$. And number of $C_4$s of the form $(u_1u_{2n+2-x}u_iu_j)$ is $(x-1)(n-1)$ where $u_i \in \{u_2,u_3,\ldots,u_x\}$ and $u_j \in \{u_2,u_3,\ldots,u_{n+1}\}\backslash\{u_i\}$.

Total number of $C_4$s of type-2 in $G_{2n+2} = \sum_{x=2}^{n}(x-1)(n-1) = (n-1)(\sum_{x=1}^{n-1} x) = \frac{n(n-1)^2}{2}$.

**Number of $C_4$s of Type-3:** In this type, $u_i \in \{u_2,u_3,\ldots,u_{n+1}\}$ and $u_j,u_k \in \{u_{n+2},u_{n+3},\ldots,u_{2n}\}$ in $G_{2n+2}$, $j \neq k$. Here, $u_j$ and $u_k$ are adjacent to $u_1$ for every $j,k \in \{n+2,n+3,\ldots,2n+1\}$ in $G_{2n+2}$, $j \neq k$. W.l.g., assume, $j<k$. If $u_j$ and $u_k$ are adjacent to $u_i$, then $j+i \leq 2n+2$ and $k+i \leq 2n+2$ which implies, $j+i<k+i \leq 2n+2$.

For $1 \leq x \leq n$, $u_{n+1+x}$ is adjacent to $u_1,u_2,\ldots,u_{n+1-x}$ in $G_{2n+2}$ and hence $d(u_{n+1+x}) = n+1-x$. In $G_{2n+2}$, $u_{n+1}$ is non-adjacent to $u_{n+2}$ and $u_{2n+1}$ is a pendant vertex and hence neither $u_{n+1}$ nor $u_{2n+1}$ is a vertex of any $C_4$ of type-3 in $G_{2n+2}$.

When $k = 2n+2-x$, $u_k = u_{2n+2-x}$ and $2 \leq x \leq n-1$, different possibilities of $u_i$ in $C_4$s of type-3 in $G_{2n+2}$are $u_2,u_3,\ldots,u_x$. And corresponding to each pair of $u_i$ and $u_k$, different possible $u_j$s are $u_{k-1},u_{k-2},\ldots,u_{n+2}$ in $G_{2n+2}$. Therefore, number of $C_4$s of type-3 in $G_{2n+2}$ with $u_k = u_{2n+2-x}$ is $(x-1)(k-1-(n+1)) = (x-1)(n-x)$. Hence, total number of $C_4$s of type-3 in $G_{2n+2} = \sum_{x=2}^{n-1}(n-x)(x-1) = \sum_{x=1}^{n-2}(n-1-x)x = (n-1)(\sum_{x=1}^{n-2} x) - \sum_{x=1}^{n-2} x^2 = \frac{n(n-1)(n-2)}{6}$.

When $u_i,u_j,u_k \in \{u_{n+2},u_{n+3},\ldots,u_{2n}\}$, cycle $C_4$ of the form $(u_1\, u_iu_ju_k)$ doesn't exist in $G_{2n+2}$ since $\{u_{n+2},u_{n+3},\ldots,u_{2n+2}\}$ is a stable set to split graph $G_{2n+2}$.

Adding all $C_4$s in the three types, we obtain, total number of $C_4$s in $G_{2n+2}$ with $u_1$ as a vertex = $\frac{n(n-1)(n-2)}{2} + \frac{n(n-1)^2}{2} + \frac{n(n-1)(n-2)}{6} = \frac{(n-1)n(7n-11)}{6}$, $2 \leq n$. Therefore, for $2 \leq n$,

$$|C_4|_{G_{2n+2}} = |C_4|_{G_{2n}} + \frac{1}{6}(7n^3-18n^2+11n)$$

$$= \frac{1}{6}((7n^3-18n^2+11\text{n}) + (7(n-1)^3-18(n-1)^2+11(n-1))) + |C_4|_{G_{2n-2}}$$

$$= \frac{1}{6}((7n^3-18n^2+11\text{n}) + (7(n-1)^3 - 18(n-1)^2 + 11(n-1)) + \ldots + (7.2^3 - 18.2^2 + 11\text{x}2)) + |C_4|_{G_4}$$

$$= \frac{1}{6}((7n^3-18n^2+11\text{n}) + (7(n-1)^3 - 18(n-1)^2 + 11(n-1)) + \ldots + (7.2^3 - 18.2^2 + 11\text{x}2)) + 0$$

$$= \frac{(n-1)n(n+1)(7n-10)}{24}.$$

Now, let us to prove the result on $G_{2n+2}^c$. Consider, graph $G_{2n+2}^c$, $n \in N$. $\{u_1,u_2,\ldots,u_n\}$ is a stable set and $\{u_{n+1},u_{n+2},\ldots,u_{2n+2}\}$ is a clique to split graph $G_{2n+2}^c$. Using Theorem 1.8, graph

$G^c_{2n+2}\backslash\{u_1,u_{2n+2}\}$ is isomorphic to $G^c_{2n}$, without the vertex labels. In $G^c_{2n+2}$, $u_{2n+2}$ is adjacent to $u_1,u_2,\ldots,u_{2n+1}$ and $u_1$ is a pendant vertex. Hence, $u_1$ is not a vertex in any cycle of length 4 in $G^c_{2n+2}$. Therefore, $|C_4|_{G^c_{2n+2}} = |C_4|_{G^c_{2n}}$ + number of cycles of length four, each with $u_{2n+2}$ as a vertex in $G^c_{2n+2}$.

Let $(u_{2n+2}\ u_ku_ju_i)$ be any cycle of length 4 in $G^c_{2n+2}$, $2 \le i,j,k \le 2n+1$ and $i,j,k$ are all different. Under the above conditions, the following three types of $C_4$s arise in $G^c_{2n+2}$. Type-1: $u_i,u_j,u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$, Type-2: $u_j,u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$ and $u_i\in\{u_2,u_3,\ldots, u_n\}$ and Type-3: $u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$ and $u_i,u_j\in\{u_2,u_3,\ldots,u_n\}$. Now, let us calculate number of $C_4$s in $G^c_{2n+2}$ in each type. W.l.g. assume that $i<j<k$.

**Number of $C_4$s of Type-1:** Here, $u_i,u_j,u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$ in $G^c_{2n+2}$. Number of ways of selecting 3 vertices $u_i,u_j,u_k$ out of $u_{n+1},u_{n+2},\ldots,u_{2n+1}$ is $(n+1)C_3$. There are 3 different $C_4$s in $G^c_{2n+2}$with $u_{2n+2}$, $u_i$, $u_j$, $u_k$ as vertices under type-1, namely, $(u_{2n+2}u_ku_ju_i)$, $(u_{2n+2}u_ku_iu_j)$ and $(u_{2n+2}u_ju_ku_i)$. Hence, total number of $C_4$s of type-1 in $G^c_{2n+2} = 3.(n+1)C_3 = \frac{(n+1)n(n-1)}{2}$.

**Number of $C_4$s of Type-2:** Here, $u_k,u_j\in\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$ and $u_i\in\{u_2,u_3,\ldots,u_n\}$. Consider all possible cycles, each of length 4 and with the vertices $u_{2n+2}$, $u_i$, $u_j$ and $u_k$in $G^c_{2n+2}$.

When $i = 2$, $u_i = u_2$ is adjacent to $u_{2n+2}$ and $u_{2n+1}$ only. And under this case, $d(u_i) = 2$, $u_k = u_{2n+1}$ and $u_j = u_{2n},u_{2n-1},\ldots,u_{n+1}$. Number of such $C_4$s is $|\{u_{2n},u_{2n-1},\ldots,u_{n+1}\}| = n$.

When $i = 3$, $u_i = u_3$ is adjacent to $u_{2n+2}$, $u_{2n+1}$ and $u_{2n}$ only and thereby $d(u_i) = 3$. And any $C_4$ of type-2 is of the form $(u_{2n+2}u_3u_{2n+1}u_x)$ or $(u_{2n+2}u_3\ u_{2n}u_y)$ where $u_x\in\{u_{2n},u_{2n-1},\ldots, u_{n+1}\}$ and $u_y\in\{u_{2n+1},u_{2n-1},u_{2n-2},\ldots,u_{n+1}\}$. Number of such $C_4$s is $2n$.

In general, when $i = x$ and $2 \le x \le n$, $u_i = u_x$is adjacent to $u_{2n+2}$, $u_{2n+1},\ldots,u_{2n+2-(x-1)}$ only and thereby $d(u_i) = x$ and number of $C_4$s of the form $(u_{2n+2}u_iu_yu_z)$ is $(x-1)n$ where $u_y\in\{u_{2n+1},u_{2n},\ldots,u_{n+1}\}$ and $u_z\in\{u_{2n+1},u_{2n},\ldots,u_{n+1}\}\backslash\{u_y\}$.

Total number of $C_4$s of type-2 in $G^c_{2n+2} = \sum_{x=2}^{n}(x-1)n = n(\sum_{x=1}^{n-1} x) = \frac{(n-1)n^2}{2}$.

**Number of $C_4$s of Type-3:** Here, $u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$ and $u_i,u_j\in\{u_2,u_3,\ldots,u_n\}$, $i\neq j$. Consider all possible cycles, each of length 4 and with the vertices $u_{2n+2}$, $u_k$, $u_j$ and $u_i$ in $G^c_{2n+2}$. For a given $i$, $2 \le i \le n-1$, $j$ takes values $i+1,i+2,\ldots,n$ and possible values of $k$ are $2n+2-1,2n+2-2,\ldots,2n+2-(i-1)$. Therefore, total number of $C_4$s of type-3 in $G^c_{2n+2} = \sum_{i=2}^{n-1}(n-i)(i-1) = \sum_{i=1}^{n-2} i(n-1-i) = (n-1)(\sum_{i=1}^{n-2} i) - \sum_{i=1}^{n-2} i^2 = \frac{(n-2)(n-1)n}{6}$.

When $u_i,u_j,u_k\in\{u_2,u_3,\ldots,u_n\}$, cycle $C_4$ of the form $(u_{2n+2}u_ku_ju_i)$ doesn't exist in $G^c_{2n+2}$ since $\{u_2,u_3,\ldots,u_n\}$ is a stable set to split graph $G^c_{2n+2}$.

Adding all $C_4$s in the three types, we obtain, total number of $C_4$s with $u_{2n+2}$ as a vertex in $G^c_{2n+2} = \frac{(n-1)n(n+1)}{2} + \frac{(n-1)n^2}{2} + \frac{(n-2)(n-1)n}{6} = \frac{(n-1)n(7n+1)}{6}$, $2 \le n$. Therefore, for $2 \le n$,

$$|C_4|_{G^c_{2n+2}} = |C_4|_{G^c_{2n}} + \frac{1}{6}(7n^3 - 6n^2 - n)$$

$$= \frac{1}{6}((7n^3 - 6n^2 - n) + (7(n-1)^3 - 6(n-1)^2 - (n-1))) + |C_4|_{G^c_{2n-2}}$$

$$= \frac{1}{6}((7n^3 - 6n^2 - n) + (7(n-1)^3 - 6(n-1)^2 - (n-1)) + \ldots + (7.2^3 - 6.2^2 - 2)) + |C_4|_{G^c_4}$$

$$= \frac{1}{6}((7n^3 - 6n^2 - n) + (7(n-1)^3 - 6(n-1)^2 - (n-1)) + \ldots + (7.1^3 - 6.1^2 - 1)).$$

$= \frac{(n-1)n(n+1)(7n+6)}{24}$, $2 \le n$. Hence the result.□

**Theorem 2.2** *For* $2 \le n$, $|C_4|_{G_{2n+3}} = |C_4|_{G_{2n+1}} + \frac{(n-1)n(7n+1)}{6} = \frac{(n-1)n(n+1)(7n+6)}{24} = |C_4|_{G^c_{2n+2}}$ *and* $|C_4|_{G^c_{2n+1}} = |C_4|_{G^c_{2n-1}} + \frac{(n-1)n(7n-11)}{6} = \frac{(n-1)n(n+1)(7n-10)}{24} = |C_4|_{G_{2n+2}}$.

**Proof:** Let $V(G_{2n+3})$ = {$u_1,u_2,\ldots,u_{2n+3}$} = $V(G^c_{2n+3})$ where $u_j$ is the vertex with sum labeling $j$ in $G_{2n+3}$ and anti-sum labeling $j$ in $G^c_{2n+3}$, $1 \le j \le 2n+3$ and $n \in N$. At first, let us to prove the result for $G_{2n+3}$, $n \in N$. {$u_1,u_2,\ldots,u_{n+2}$} is a clique and {$u_{n+3},u_{n+4},\ldots,u_{2n+3}$} is a stable set to $G_{2n+3}$. Using Theorem 1.6, graph $G_{2n+3}\backslash\{u_1,u_{2n+3}\}$ is isomorphicto $G_{2n+1}$, without the vertex labels. Also, in $G_{2n+3}$, $u_1$ is adjacent to $u_2,u_3,\ldots,u_{2n+2}$; $u_{2n+3}$ is an isolated vertex and $u_{2n+2}$ is a pendant vertex. Therefore, $|C_4|_{G_{2n+3}} = |C_4|_{G_{2n+1}}$ + number of cycles of length four, each with $u_1$ as a vertex in $G_{2n+3}$. Also, none of $u_{2n+2}$ and $u_{2n+3}$ is a vertex of any cycle of length 4 in $G_{2n+3}$.

Let $(u_1\, u_i u_j u_k)$ be any cycle of length 4 (with $u_1$ as a vertex) in $G_{2n+3}$, $1 < i,j,k < 2n+2$ and $i,j,k$ are all different. Under the above conditions, the following three types of $C_4$s arise in $G_{2n+3}$. Type-1: $u_i,u_j,u_k \in \{u_2,u_3,\ldots,u_{n+2}\}$, Type-2: $u_i,u_j \in \{u_2,u_3,\ldots,u_{n+2}\}$ and $u_k \in \{u_{n+3},u_{n+4},\ldots,u_{2n+1}\}$ and Type-3: $u_i \in \{u_2,u_3,\ldots,u_{n+2}\}$ and $u_j,u_k \in \{u_{n+3},u_{n+4},\ldots,u_{2n+1}\}$. Now, let us calculate number of $C_4$s in $G_{2n+3}$ in each type.

**Number of $C_4$s of Type-1:** Here , $u_i,u_j,u_k \in \{u_2,u_3,\ldots,u_{n+2}\}$ in $G_{2n+3}$. Number of ways of selecting 3 vertices $u_i,u_j,u_k$ out of $u_2,u_3,\ldots,u_{n+2}$ is $(n+1)C_3$. There are 3 different $C_4$s with $u_1$, $u_i$, $u_j$, $u_k$ as vertices under Type-1, namely, $(u_1u_iu_ju_k)$, $(u_1u_iu_ku_j)$ and $(u_1u_ju_iu_k)$. Therefore, total number of $C_4$s of type-1 in $G_{2n+3}$ = $3.(n+1)C_3$ = $\frac{n(n-1)(n-2)}{2}$.

**Number of $C_4$s of Type-2:** Here, $u_i,u_j \in \{u_2,u_3,\ldots,u_{n+2}\}$ and $u_k \in \{u_{n+},u_{n+4},\ldots,u_{2n+1}\}$. Consider all possible cycles, each of length 4 and with vertices $u_1$, $u_i$, $u_j$ and $u_k$in $G_{2n+3}$.

When $k = 2n+1$, $u_k = u_{2n+1}$ is adjacent to $u_1$ and $u_2$ only. And under this case, $u_2 = u_i$ or $u_2 = u_j$. W.l.g., assume $u_2 = u_i$. This implies, $2 = i < 3 \le j \le n+2$. And any $C_4$ under this case is of the form $(u_1u_{2n+1}u_2u_j)$, $u_j \in \{u_3,u_4,\ldots,u_{n+2}\}$ and number of such $C_4$s is $n$.

When $k = 2n$, $u_k = u_{2n}$ is adjacent to $u_1$, $u_2$ and $u_3$ only. And any $C_4$ of type-2 is of the form $(u_1u_{2n}u_2u_x)$ or $(u_1u_{2n}u_3u_y)$, $u_x \in \{u_3,u_4,\ldots,u_{n+2}\}$ and $u_y \in \{u_2,u_4,u_5,\ldots,u_{n+2}\}$. Number of such $C_4$s is $2n$.

When $k = 2n-1$, $u_k = u_{2n-1}$ is adjacent to $u_1$ ,$u_2$ , $u_3$ and $u_4$ only and thereby $d(u_k) = 4$. Therefore, number of such $C_4$s is $(4-1)n = 3n$.

In general, when $k = 2n+3-x$ and $2 \le x \le n$, $u_k = u_{2n+3-x}$is adjacent to $u_1$ , $u_2$ , . . . , $u_x$ and thereby $d(u_k) = d(u_{2n+3-x}) = x$ and number of $C_4$s of the form $(u_1u_{2n+3-x}u_iu_j)$ in $G_{2n+3}$ is $(x-1)n$ where $u_i \in \{u_2,u_3,\ldots,u_x\}$ and $u_j \in \{u_2,u_3,\ldots,u_{n+2}\}\backslash\{u_i\}$. Therefore, total number of $C_4$s of type-2 in $G_{2n+3}$ = $\sum_{x=2}^{n}(x-1)n = n(\sum_{x=1}^{n-1}x) = \frac{(n-1)n^2}{2}$.

**Number of $C_4$s under Type-3:** Here, $u_i \in \{u_2,u_3,\ldots,u_{n+2}\}$ and $u_j,u_k \in \{u_{n+3},u_{n+4},\ldots,u_{2n+1}\}$ and $u_j$ and $u_k$ are adjacent to $u_1$ for every $j,k \in \{n+3,n+4,\ldots,2n+2\}$ in $G_{2n+3}$, $j \ne k$. W.l.g., assume, $j<k$. If $u_j$ and $u_k$ are adjacent to $u_i$, then $j+i<k+i \le 2n+3$.

In $G_{2n+3}$, $u_{n+2+x}$ is adjacent to $u_1,u_2,\ldots,u_{n+1-x}$, $1 \le x \le n$ and thereby $d(u_{n+2+x}) = n+1-x$. Also, $u_{n+1}$ and $u_{n+2}$ are non-adjacent to $u_{n+3}$ and $u_{2n+2}$ is a pendant vertex. Hence, none of $u_{n+1}$, $u_{n+2}$ and $u_{2n+2}$ is a vertex of any $C_4$ of type-3 in $G_{2n+3}$.

When $u_k = u_{2n+3-x}$ and $2 \le x \le n-1$, different possibilities of $u_i$ in $C_4$s of type-3 in $G_{2n+3}$are $u_2$, $u_3$, . . . ,$u_x$. And corresponding to each pair of $u_i$ and $u_k$, different possibilities of $u_j$ are $u_{k-1},u_{k-2},\ldots,u_{n+3}$ in $G_{2n+3}$. Therefore, number of $C_4$s of type-3 in $G_{2n+3}$ with $u_k = u_{2n+3-x}$ is $(x-1)(k-1-(n+2)) = (x-1)(n-x)$. Hence, total number of $C_4$s of type-3 in $G_{2n+3}$ = $\sum_{x=2}^{n-1}(n-x)(x-1) = \sum_{x=1}^{n-2}(n-1-x)x = \frac{n(n-1)(n-2)}{6}$.

Cycle $C_4$ of the form $(u_1u_iu_ju_k)$ with $u_i,u_j,u_k \in \{u_{n+3},u_{n+4},\ldots,u_{2n+3}\}$ doesn't exist in $G_{2n+3}$ since $\{u_{n+3},u_{n+4},\ldots,u_{2n+3}\}$ is a stable set to split graph $G_{2n+3}$.

Adding all $C_4$s in the three types, we obtain, total number of $C_4$s in $G_{2n+3}$ with $u_1$ as a vertex = $\frac{(n+1)n(n-1)}{2} + \frac{n^2(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{(n-1)n(7n+1)}{6}$, $2 \le n$. Therefore, for $2 \le n$,

$$|C_4|_{G_{2n+3}} = |C_4|_{G_{2n+1}} + \frac{1}{6}(7n^3 - 6n^2 - n)$$

$=\frac{1}{6}((7n^3\text{-}6n^2-n)+(7(n-1)^3\text{-}6(n-1)^2-(n-1)))+|C_4|_{G_{2n-1}}$

$=\frac{1}{6}((7n^3\text{-}6n^2-n)+(7(n-1)^3-6(n-1)^2-(n-1))+\ldots+(7.2^3\text{-}6.2^2-2))+|C_4|_{G_4}$

$=\frac{1}{6}((7n^3\text{-}6n^2-n)+(7(n-1)^3-6(n-1)^2-(n-1))+\ldots+(7.2^3\text{-}6.2^2-2))+0$

$=\frac{(n-1)n(n+1)(7n+6)}{24}$.

Now, let us prove the result on $G^c_{2n+1}$. Consider graph $G^c_{2n+1}$, $n\in N$. $\{u_1,u_2,\ldots,u_n\}$ is a stable set and $\{u_{n+1},u_{n+2},\ldots,u_{2n+1}\}$ is a clique to $G^c_{2n+1}$. Using Theorem 1.6, graph $G^c_{2n+1}\backslash\{u_1,u_{2n+1}\}$ is isomorphic to $G^c_{2n-1}$, without the vertex labels. In $G^c_{2n+1}$, $u_{2n+1}$ is adjacent to $u_1,u_2,\ldots,u_{2n}$ and $u_1$ is a pendant vertex. Hence, $u_1$ is not a vertex in any cycle of length 4 in $G^c_{2n+1}$. Therefore, $|C_4|_{G^c_{2n+1}}=|C_4|_{G^c_{2n-1}}+$ number of cycles of length four, each with $u_{2n+1}$ as a vertex in $G^c_{2n+1}$.

Let $(u_{2n+1}u_ku_ju_i)$ be any cycle of length 4 with $u_{2n+1}$ as a vertex in $G^c_{2n+1}$, $2\leq i,j,k\leq 2n$ and $i,j,k$ are all different. Under the above conditions, the following three types of $C_4$s with $u_{2n+1}$ as a vertex arise in $G^c_{2n+1}$. Type-1: $u_i,u_j,u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$, Type-2: $u_j,u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$ and $u_i\in\{u_2,u_3,\ldots,u_n\}$ and Type-3: $u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$ and $u_i,u_j\in\{u_2,u_3,\ldots,u_n\}$. Now, let us calculate number of $C_4$s in $G^c_{2n+1}$ in each type. W.l.g., assume that $i<j<k$.

**Number of $C_4$s under Type-1:** Here, $u_i,u_j,u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$ in $G^c_{2n+1}$. Number of ways of selecting 3 vertices $u_i,u_j,u_k$ out of $u_{n+1},u_{n+2},\ldots,u_{2n}$ is $nC_3$. There are 3 different $C_4$s with $u_{2n+1}$, $u_i$, $u_j$, $u_k$ as vertices under type-1, namely, $(u_{2n+1}u_ku_ju_i)$, $(u_{2n+1}u_ku_iu_j)$ and $(u_{2n+1}u_ju_ku_i)$. Hence, total number of $C_4$s of type-1 in $G^c_{2n+1}=3.nC_3=\frac{n(n-1)(n-2)}{2}$.

**Number of $C_4$s under Type-2:** Here, $u_k,u_j\in\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$ and $u_i\in\{u_2,u_3,\ldots,u_n\}$. Consider all possible cycles $(u_{2n+1}u_iu_ju_k)$ in $G^c_{2n+1}$.

When $i=2$, $u_i=u_2$ is adjacent to $u_{2n+1}$ and $u_{2n}$ only. And under this case, $d(u_i)=2$, $u_k=u_{2n}$ and $u_j=u_{2n-1},u_{2n-2},\ldots,u_{n+1}$. Number of such $C_4$s is $|\{u_{2n-1},u_{2n-2},\ldots,u_{n+1}\}|=n\text{-}1$.

When $i=3$, $u_i=u_3$ is adjacent to $u_{2n+1}$, $u_{2n}$ and $u_{2n-1}$ only and $d(u_i)=3$. And any $C_4$ of type-2 is of the form $(u_{2n+1}u_3u_{2n}u_x)$ or $(u_{2n+1}u_3u_{2n-1}u_y)$, $u_x\in\{u_{2n-1},u_{2n-2},\ldots,u_{n+1}\}$ and $u_y\in\{u_{2n},u_{2n-2},u_{2n-3},\ldots,u_{n+1}\}$. Number of such $C_4$s is $2(n\text{-}1)$.

In general, when $i=x$ and $2\leq x\leq n$, $u_i=u_x$is adjacent to $u_{2n+1}$, $u_{2n},\ldots,u_{2n+1-(x-1)}$ only and thereby $d(u_i)=x$ and $u_k$ takes values $u_{2n},u_{2n-1},\ldots,u_{2n+1-(x-1)}$ and $u_j\in\{u_{2n},u_{2n-1},\ldots,u_{n+1}\}\backslash\{u_k\}$. Therefore, number of $C_4$s of the form $(u_{2n+1}u_iu_ku_j)$ is $(x\text{-}1)(n\text{-}1)$, $2\leq x\leq n$. Here, $j$ need not be less than $k$.

Total number of $C_4$s of type-2 in $G^c_{2n+1}=\sum_{x=2}^{n}(x-1)(n-1)=(n-1)(\sum_{x=1}^{n-1}x)=\frac{(n-1)^2n}{2}$.

**Number of $C_4$s under Type-3:** Here, $u_k\in\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$ and $u_i,u_j\in\{u_2,u_3,\ldots,u_n\}$, $i\neq j$. Consider all possible cycles $(u_{2n+1}u_iu_ju_k)$ in $G^c_{2n+1}$. For a given $i$, $2\leq i\leq n\text{-}1$, $j$ takes values $i+1,i+2,\ldots,n$ and possible values of $k$ are $2n1,2n\text{-}1,\ldots,2n\text{-}(i\text{-}2)$. Hence, total number of $C_4$s of type-3 in $G^c_{2n+1}=\sum_{i=2}^{n-1}(n-i)(i-1)=\sum_{i=1}^{n-2}i(n-1-i)=\frac{(n-2)(n-1)^2}{6}-\frac{(n-2)(n-1)(2n-3)}{6}=\frac{(n-2)(n-1)n}{6}$.

Cycle $C_4$ of the form $(u_{2n+1}u_ku_ju_i)$ with $u_i,u_j,u_k\in\{u_2,u_3,\ldots,u_n\}$ doesn't exist in $G^c_{2n+1}$ since $\{u_2,u_3,\ldots,u_n\}$ is a stable set to split graph $G^c_{2n+1}$.

Adding all $C_4$s in the three types, we obtain, total number of $C_4$s with $u_{2n+1}$ as a vertex in $G^c_{2n+1}=\frac{(n-2)(n-1)n}{2}+\frac{(n-1)^2n}{2}+\frac{(n-2)(n-1)n}{6}=\frac{(n-1)n(7n-11)}{6}$, $2\leq n$. Therefore, for $2\leq n$,

$|C_4|_{G^c_{2n+1}}=|C_4|_{G^c_{2n-1}}+\frac{1}{6}(7n^3\text{-}18n^2+11n)$

$=\frac{1}{6}((7n^3\text{-}18n^2+11n)+(7(n-1)^3\text{-}18(n-1)^2+11(n-1)))+|C_4|_{G^c_{2n-2}}$

$=\frac{1}{6}((7n^3\text{-}18n^2+11n)+(7(n-1)^3\text{-}18(n-1)^2+11(n-1)))+\ \ .\ \ .\ \ .\ \ +$
$(7.2^3\text{-}18.2^2+11(2)))+|C_4|_{G^c_3}$

$=\frac{1}{6}((7n^3$-$18n^2+11n)$+$(7(n-1)^3$-$18(n-1)^2+11(n-1)))$ +...+$(7.1^3$-$18.1^2$+11)).

$=\frac{(n-1)n(n+1)(7n-10)}{24}$. The rest of the result follows from Theorem 2.1. □

**Lemma 2.3** Let $V(G_n)$ = {$v_1,v_2,\ldots,v_n$} = $V(G_n^c)$ *where $v_j$ is the vertex with integral sum labeling j in $G_n$ and anti-integral sum labeling j in $G_n^c$, 1 ≤ j ≤ n and n∈N. Then,* (i)$G_{0,n}\cong G_{n+2}\setminus\{v_{n+2}\}$, (ii) $G_{n+2}\cong(G_n+\{v_{n+1}\})\cup\{v_{n+2}\}$, (iii) $G_{n+2}^c\cong(G_n^c\cup\{v_{n+1}\})+\{v_{n+2}\}$*and* (iv) $G_{-1,n}\cong G_{n+4}\setminus\{v_{n+3},v_{n+4}\}$, *without the vertex labels.*

**Proof :(i)** We have $G_{-m,n}=K_1+((-G_m)+G_n)$, $m,n\in N_0$. Therefore, $G_{0,n}=K_1+G_n$, $n\in N$. Let $V(G_{0,n})$ = {$u_0,u_1,u_2,\ldots,u_n$} and $V(G_{n+2})$ = {$v_1,v_2,\ldots,v_{n+2}$} where $u_i$ is the vertex with integral sum labeling $i$ for $i$ = 0,1,…,$n$ and $v_j$ is the vertex of $G_{n+2}$ with integral sum labeling $j$, 1 ≤ $j$ ≤ $n$+2. Define $f$: $V(G_{0,n})\to V(G_{n+2}\setminus\{v_{n+2}\})$ such that $f(u_i)=v_{i+1}$ and $f((u,v))=(f(u),f(v))$ for every $(u,v)\in E(G_{0,n})$, $i$ = 0,1,…,$n$. Now, $(u_x,u_y)\in E(G_{0,n})$ if and only if 0 <$x$+$y$<$n$+1 if and only if 2 < ($x$+1)+($y$+1) <$n$+3 if and only if $(v_{x+1},v_{y+1})=(f(u_x),f(u_y))\in E(G_{n+2})=E(G_{n+2}\setminus\{v_{n+2}\})$. This implies, $f$ is a bijective mapping and preserves adjacency. Hence, $G_{0,n}\cong G_{n+2}\setminus\{u_{n+2}\}$, without the vertex labels.

(ii) Using (i), we obtain, $G_{n+2}\cong G_{0,n}\cup\{v_{n+2}\}\cong(G_n+K_1)\cup\{v_{n+2}\}\cong(G_n+\{v_{n+1}\})\cup\{v_{n+2}\}$, without the vertex labels, $n\in N$.

(iii) Using (ii), we obtain, $G_{n+2}^c\cong((G_n+\{v_{n+1}\})\cup\{v_{n+2}\})^c\cong(G_n+\{v_{n+1}\})^c+\{v_{n+2}\}\cong(G_n^c\cup\{v_{n+1}\})+\{v_{n+2}\}$, without the vertex labels, $n\in N$.

(iv) We have $G_{-1,n}=K_1+((-K_1)+G_n)=K_1+G_{0,n}\cong K_1+(G_{n+2}\setminus\{u_{n+2}\})$, without the vertex labels, using (i), $n\in N$. Let $V(G_{-1,n})$ = {$u_0,u_1,u_2,\ldots,u_{n+1}$} and $V(G_{n+4})$ = {$v_1,v_2,\ldots,v_{n+4}$} where $u_i$ is the vertex with integral sum labeling $i$ for $i$ = 0,1,…,$n$ and $u_{n+1}$ is the vertex with integral sum labeling -1 in $G_{-1,n}$ and $v_j$ is the vertex of $G_{n+4}$ with integral sum labeling $j$, 1 ≤ $j$ ≤ $n$+4. Using Theorem 1.6, graph $G_{n+4}\setminus\{v_1,v_2,v_{n+3},v_{n+4}\}$ is isomorphic to $G_n$, without the vertex labels. And so $((G_{n+4}\setminus\{v_{n+4},v_{n+3},v_2,v_1\})+K_1)+K_1\cong G_{-1,n}$without the vertex labels. Define $f$ : $V(G_{-1,n})\to V(G_{n+4}\setminus\{v_{n+4},v_{n+3}\})$ such that $f(u_0)=v_1$, $f(u_{n+1})=v_2$, $f(u_i)=v_{i+2}$ for $i$ = 1,2,…,$n$ and $f((u,v))=(f(u),f(v))$ for every $(u,v)\in E(G_{-1,n})$. Now, let us consider images of edges incident at each point $u_0$ and $u_{n+1}$, separately. In $G_{-1,n}$, integral sum labeling of $u_0$ and $u_{n+1}$ are 0 and -1, respectively, $u_0$ and $u_{n+1}$ are adjacent and each one is adjacent to $u_j$ for $j$ = 1,2,…,$n$. Now, $f((K_1(0),u_i))=f((u_0,u_i))=(f(u_0),f(u_i))=(v_1,v_{i+1})\in E(G_{n+4}\setminus\{v_{n+3},v_{n+4}\})$ for every $i$ since 1+($i$+1) ≤ $n$+2, $i$ = 1,2,…,$n$; $f((K_1(0),u_{n+1}))=f((u_0,u_{n+1}))=(f(u_0),f(u_{n+1}))=(v_1,v_2)\in E(G_{n+4}\setminus\{v_{n+3},v_{n+4}\})$ and $f((u_{n+1},u_j))=(f(u_{n+1}),f(u_j))=(v_2,v_{j+2})\in E(G_{n+4}\setminus\{v_{n+3},v_{n+4}\})$ for every $j$, $j$ = 1,2,…,$n$. Therefore, $f$ is a bijective mapping preserving adjacency and hence, $G_{-1,n}\cong G_{n+4}\setminus\{v_{n+3},v_{n+4}\}$, without the vertex labels. □

**Result 2.4 [Algorithm to generate $G_n$and $G_n^c$]**

Starting with either$\boldsymbol{G_0}$or$\boldsymbol{G_1}$and using results (ii) and (iii) of Lemma 2.3 for $n$ = 2,4,… or $n$ = 3,5,…, one can generate sum graphs$\boldsymbol{G_n}$and anti-sum graphs$\boldsymbol{G_n^c}$of any order without using definitions of sum and anti-sum labeling.

**Theorem 2.5** *Forn∈N,*$|E(G_{0,n})|=|E(G_{n+2})|=n+|E(G_n)|$, $|E(G_{-1,n})|=|E(G_{n+4})|-1=2n+1+|E(G_n)|$, $|C_3|_{G_{0,n}}=|C_3|_{G_{n+2}}$, $|C_3|_{G_{-1,n}}=|C_3|_{G_{n+4}}$, $|C_4|_{G_{0,n}}=|C_4|_{G_{n+2}}$*and*$|C_4|_{G_{-1,n}}=|C_4|_{G_{n+4}}$.

Proof: Result follows from Lemma 2.3. □

**Theorem 2.6** *Forn∈N,* $|C_4|_{G_{0,2n}}=|C_4|_{G_{2n+2}}=\frac{(n-1)n(n+1)(7n-10)}{24}$, $|C_4|_{G_{0,2n+1}}=|C_4|_{G_{2n+3}}=\frac{(n-1)n(n+1)(7n+6)}{24}$, $|C_4|_{G_{-1,2n}}=|C_4|_{G_{2n+4}}=\frac{n(n+1)(n+2)(7n-3)}{24}$*and* $|C_4|_{G_{-1,2n+1}}=|C_4|_{G_{2n+5}}=\frac{n(n+1)(n+2)(7n+13)}{24}$.

Proof:Result follows from Theorems 2.2 and 2.5. □

**Theorem 2.7** *Number of$P_3$s in* $G_{2n}$ *such that each* $P_3$ = *uvwwithuw*$\notin E(G_{2n})$ *is*$\frac{(n-1)n(2n-1)}{6}$, *u,v,w*$\in V(G_{2n})$ *and number of* $P_3$*s in*$G_{2n+1}$*such that each* $P_3$ = *uvw with uw*$\notin E(G_{2n})$ *is*$\frac{(n-1)n(n+1)}{3}$, *u,v,w*$\in V(G_{2n+1})$ *and n*$\in N$.

**Proof:** Let $V(G_{2n}) = \{u_1,u_2,\ldots,u_{2n}\}$ where $u_j$ is the vertex of $G_{2n}$ with sum labeling $j$, $j = 1,2,\ldots,2n$. $\{u_1,u_2,\ldots,u_n\}$ is a clique and $\{u_{n+1},u_{n+2},\ldots,u_{2n}\}$ is a stable set to split graph $G_{2n}$ and vertex $u_n$ is non-adjacent to $u_{n+1},u_{n+2},\ldots,u_{2n}$. Each required $P_3$ in $G_{2n}$ contains at least one element of $\{u_{n+1},u_{n+2},\ldots,u_{2n-1}\}$. In $G_{2n}$, counting of $P_3$s such that each $P_3 = uvw$ and $uw\notin E(G_{2n})$ is done as follows, $u,v,w\in V(G_{2n})$. W.l.g., assume that $1 \leq i<j< 2n-k \leq 2n-1$. For $1 \leq k \leq n-1$, vertex $u_{2n-k}$ is adjacent to $v_i$ for $i = 1,2,\ldots,k$ and $P_3 = u_{2n-k}u_iu_j$ is a required path on the 3 vertices for $j = k+1,k+2,\ldots,2n-k-1$. Therefore, in $G_{2n}$, number of $P_3$s such that each $P_3 = uvw$ with $uw\notin E(G_{2n})$ and $u,v,w\in V(G_{2n})$ = $\sum_{k=1}^{n-1}(\sum_{i=1}^{k}(2n-2k-1))$ = $\sum_{k=1}^{n-1}k(2n-1-2k)$ = $\frac{(n-1)n(2n-1)}{2}$ - $\frac{(n-1)n(2n-1)}{3} = \frac{(n-1)n(2n-1)}{6}$.

Similarly, let $V(G_{2n+1})= \{u_1,u_2,\ldots,u_{2n}\}$ where $u_j$ is the vertex of $G_{2n+1}$ with sum labeling $j$, $j = 1,2,\ldots,2n+1$. $\{u_1,u_2,\ldots,u_{n+1}\}$ is a clique and $\{u_{n+2},u_{n+3},\ldots,u_{2n+1}\}$ is a stable set to split graph $G_{2n+1}$ and vertex $u_{n+1}$ is non-adjacent to $u_{n+2},u_{n+3},\ldots,u_{2n+1}$. Each required $P_3$ in $G_{2n+1}$ contains at least one element of $\{u_{n+2},u_{n+3},\ldots,u_{2n}\}$. In $G_{2n+1}$, counting of $P_3$s such that each $P_3 = uvw$ and $uw\notin E(G_{2n+1})$ is done as follows, $u,v,w\in V(G_{2n+1})$. W.l.g., assume that $1 \leq i<j< 2n-k \leq 2n$. For $1 \leq k \leq n-1$, vertex $u_{2n+1-k}$ is adjacent to $v_i$ for $i = 1,2,\ldots,k$ and $P_3 = u_{2n+1-k}u_iu_j$ is a required path on the 3 vertices for $j = k+1,k+2,\ldots,2n+1-k-1$. Therefore, in $G_{2n+1}$, number of $P_3$s such that each $P_3 = uvw$ with $uw\notin E(G_{2n+1})$ and $u,v,w\in V(G_{2n+1})$ = $\sum_{k=1}^{n-1}(\sum_{i=1}^{k}(2n-2k))$ = $\sum_{k=1}^{n-1}k(2n-2k)$ = $\frac{2n(n-1)n}{2}$ - $\frac{2(n-1)n(2n-1)}{6} = \frac{(n-1)n(n+1)}{3}$. Hence the result. □

**Theorem 2.8** *For* $2 \leq m,n$, (i) $|C_4|_{G_{-m,n}} = |C_4|_{G_m} + |C_4|_{G_n}$ + *mC*$_2$.*nC*$_2$ + *number of* $C_4$s *with*$K_1$*as a vertex in* $G_{-m,n}$ = $|C_4|_{G_m} + |C_4|_{G_n}$ + 3($|C_3|_{G_m} + |C_3|_{G_n}$) + 2($n.|E(-G_m)| + m.|E(G_n)|$) + *mC*$_2$.*nC*$_2$ + *n.mC*$_2$ + *m.nC*$_2$ + (*number of*$P_3$*s in* -$G_m$, *each* $P_3$ = *uvwwithuw*$\notin E(-G_m)$) + (*number of*$P_3$*s in*$G_n$, *each*$P_3$ = *uvwwithuw*$\notin E(G_n)$) *and* (ii) $|C_4|_{G^c_{-m,n}} = |C_4|_{G^c_m} + |C_4|_{G^c_n}$.

**Proof:** We have $G_{-m,n} = K_1 + ((-G_m) + G_n) = K_1 + ((-G_m) \cup G_n \cup K_{m,n})$ and $G^c_{-m,n} = K_1(0) \cup ((-G^c_m) \cup G^c_n)$ where the vertices of $K_{m,n}$are vertices of $(-G_m)\cup G_n$, $m,n\in N_0$. Here, $K_1$ is the vertex with integral sum label 0 and adjacent to all other vertices in $G_{-m,n}$ and an isolated vertex in $G^c_{-m,n}$.Clearly, $|C_4|_{G^c_{-m,n}} = |C_4|_{G^c_{-m}} + |C_4|_{G^c_n}$ since $G^c_{-m}$and $G^c_n$ are disjoint subgraphs in $G^c_{-m,n}$. Now, $C_4$s in $(-G_m\cup G_n\cup K_{m,n})$ = ($C_4$s in $-G_m$)$\cup$ ($C_4$s in $G_n$) $\cup$ ($C_4$s in $K_{m,n}$) and$|C_4|_{G_{-m,n}}$= number of $C_4$s, each $C_4$ with $K_1$ as a vertex in $G_{-m,n}$ + number of $C_4$s, each $C_4$ without $K_1$ as a vertex in $G_{-m,n}$ = $|C_4|_{G_m} + |C_4|_{G_n} + |C_4|_{K_{m,n}}$ + number of $C_4$s with $K_1$ as a vertex in $G_{-m,n}$ = $|C_4|_{G_m} + |C_4|_{G_n}$ + *mC*$_2$.*nC*$_2$ + number of $C_4$s with $K_1$ as a vertex in $G_{-m,n}$ since $K_{m,n}$ is a complete bipartite graph and number of $C_4$s in $K_{m,n}$ is *mC*$_2$.*nC*$_2$, $2 \leq m,n$..

Let $V(G_{-m,n}) = \{u_0,u_1,u_2,\ldots,u_{m+n}\}$ where, in $G_{-m,n}= K_1+ ((-G_m)+ G_n)$, $u_0$ is the vertex $K_1$ with integral sum labeling 0, $u_i$ is the vertex of $-G_m$ with integral sum labeling $-i$ for $i = 1,2,\ldots,m$ and $u_{m+j}$ is the vertex of $G_n$ with integral sum labeling $j$, $j = 1,2,\ldots,n$. Let $1 \leq |i| <|j| <|k| \leq m+n$ and $(u_0u_iu_ju_k)$ be any cycle of length 4 with $u_0$ as a vertex in $G_{-m,n}$. The following types of $C_4$s with $u_0$ as a vertex arise. Type-1: $u_i,u_j,u_k\in V(-G_m)$; Type-2: $u_i,u_j,u_k\in V(G_n)$; Type-3: $u_i,u_j\in V(-G_m)$ and $u_k\in V(G_n)$ and Type-4: $u_i\in V(-G_m)$ and $u_j,u_k\in V(G_n)$. Let us obtain number of $C_4$s with $K_1$ as a vertex in $G_{-m,n}$ in each type.

**Number of $C_4$s under Type-1:** Here, $u_i,u_j,u_k\in V(-G_m)$. In this case, $C_4$ is formed in $G_{-m,n}$ with vertices $u_0$, $u_i$, $u_j$ and $u_k$, either $(u_iu_ju_k)$ is a cycle of length 3 in $-G_m$ or $u_iu_ju_k$ is a path of length 2 in $G_{-m}$ with $u_iu_k\notin E(-G_m)$. When $(u_iu_ju_k)$ is a cycle of length 3 in $-G_m$, possible type-1 $C_4$s in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ are $(u_0u_iu_ju_k)$, $(u_0u_iu_ku_j)$ and $(u_0u_ju_iu_k)$.Hence, number of $C_4$s in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ when $(u_iu_ju_k)$ is a cycle of length 3 in $-G_m$ = 3.$|C_3|_{G_m}$. Similarly, when $u_iu_ju_k$ is a path of length 2 in $-G_m$ and $u_iu_k\notin E(-G_m)$, then the only possible type-1 $C_4$ in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ is $(u_0u_iu_ju_k)$.Thus, number of $C_4$s in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ when $u_iu_ju_k$ is a path of length 2 in -$G_m$ but $u_iu_k$ is not an edge of -$G_m$ = number of $P_3$s in -$G_m$, each $P_3$ is not a subgraph of any $C_3$ of -$G_m$. Hence, number of $C_4$s of type-1 in $G_{-m,n}$ = 3.$|C_3|_{G_m}$ + number of $P_3$s in -$G_m$ such that each $P_3$ is not a subgraph of any $C_3$ of -$G_m$.

**Number of $C_4$s under Type-2:** Here, $u_i,u_j,u_k \in V(G_n)$. Similar to type-1 and we obtain, number of $C_4$s of type-2 in $G_{-m,n}$ = 3.$|C_3|_{G_n}$ + number of $P_3$s in $G_n$ such that each $P_3$ is not a subgraph of any $C_3$ of $G_n$.

**Number of $C_4$s under Type-3:** Here, $u_i,u_j \in V(-G_m)$ and $u_k \in V(G_n)$. In this case, $C_4$ is formed in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ such that either $u_i$ and $u_j$ are adjacent or $u_i$ and $u_j$ are non-adjacent whereas $u_k$ takes all vertices of $G_n$. When $u_i$ and $u_j$ are adjacent, possible $C_4$s of type-3 in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ are $(u_0u_iu_ju_k)$, $(u_0u_iu_ku_j)$ and $(u_0u_ju_iu_k)$.Therefore, number of $C_4$s of type-3 in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ when $u_i$ and $u_j$ are adjacent = 3.$|E(-G_m)|$.(number of vertices of $G_n$) = $3n.|E(-G_m)|$. Similarly, when $u_i$ and $u_j$ are non-adjacent, the only possible type-3 $C_4$ in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ is $(u_0\, u_iu_ku_j)$.Number of non-adjacent pair of vertices in -$G_m$ = $mC_2$ -number of adjacent pair of vertices in -$G_m$ = $mC_2$ - $|E(G_{-m})|$. Hence, number of $C_4$s of type-3 in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ when $u_i$ and $u_j$ are non-adjacent = $n(mC_2 - |E(-G_m)|)$. Therefore, number of $C_4$s of type-3 in $G_{-m,n}$ = number of $C_4$s of type-3 in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ when $u_i$ and $u_j$ are adjacent + number of $C_4$s of type-3 in $G_{-m,n}$ with vertices $u_0,u_i,u_j,u_k$ when $u_i$ and $u_j$ are non-adjacent = $n(mC_2 + 2.|E(-G_m)|)$.

**Number of $C_4$s under Type-4:** Here, $u_i \in V(-G_m)$ and $u_j,u_k \in V(G_n)$. Similarly, we obtain, number of $C_4$s of type-4 in $G_{-m,n}$ = $m(nC_2 + 2.|E(G_n)|)$. Therefore, for $2 \le m,n$,

Number of $C_4$s in $G_{-m,n}$= $|C_4|_{G_{-m,n}}$

= number of $C_4$s of type-1 in $G_{-m,n}$ + number of $C_4$s of type-2 in $G_{-m,n}$ + number of $C_4$s of type-3 in $G_{-m,n}$ + number of $C_4$s of type-4 in $G_{-m,n}$

= $|C_4|_{G_m} + |C_4|_{G_n} + mC_2.nC_2 + 3|C_3|_{G_m} + 3|C_3|_{G_n}$

+ number of $P_3$s in -$G_m$ such that each $P_3$ is not a subgraph of any $C_3$ of -$G_m$

+ number of $P_3$s in $G_n$ such that each $P_3$ is not a subgraph of any $C_3$ of $G_n$

+ $n(mC_2 + 2|E(G_m)|) + m(nC_2 + 2|E(G_n)|)$. Hence the result. □

**Corollary 2.9** *For* $m,n \in N$,

(i) $|C_4|_{G_{-2m,2n}} = \frac{(m-1)m(7m^2+m-18)}{24} + \frac{(n-1)n(7n^2+n-18)}{24} + mn(4mn + 6(m+n) - 11)$;

(ii) $|C_4|_{G_{-2m,2n+1}} = \frac{(m-1)m(7m^2+m-18)}{24} + \frac{(n-1)n(7n^2+17n-2)}{24} +$
$+ m(4m-3)(2n+1) + mn(4mn+2m+6n+1)$;

(iii) $|C_4|_{G_{-(2m+1),2n}} = \frac{(m-1)m(7m^2+17m-2)}{24} + \frac{(n-1)n(7n^2+n-18)}{24}$
$+ (2m+1)n(4n-3) + mn(4mn+6m+2n+1)$;

(iv) $|C_4|_{G_{-(2m+1),2n+1}} = \frac{(m-1)m(7m^2+17m-2)}{24} + \frac{(n-1)n(7n^2+17n-2)}{24}$
$+ (mn+m+n)(2m+1)(2n+1) + 4mn(m+n) + 2(m^2+n^2)$;

(v) $|C_4|_{G^c_{-2m,2n}} = \frac{(m-2)(m-1)m(7m-1)}{24} + \frac{(n-2)(n-1)n(7n-1)}{24}$;

(vi) $|C_4|_{G^c_{-2m,2n+1}} = \frac{(m-2)(m-1)m(7m-1)}{24} + \frac{(n-1)n(n+1)(7n-10)}{24}$;

(vii) $|C_4|_{G^c_{-(2m+1),2n}} = \frac{(m-1)m(m+1)(7m-10)}{24} + \frac{(n-2)(n-1)n(7n-1)}{24}$ *and*

(viii) $|C_4|_{G^c_{-(2m+1),2n+1}} = \frac{(m-1)m(m+1)(7m-10)}{24} + \frac{(n-1)n(n+1)(7n-10)}{24}$.

**Proof:** For $m,n\in N$, using Theorems 1.5, 2.1, 2.2, 2.7, 2.8 and Corollary 1.11, we obtain, (i)
$|C_4|_{G_{-2m,2n}} = |C_4|_{G_{2m}} + |C_4|_{G_{2n}} + 3(|C_3|_{G_{2m}} + |C_3|_{G_{2n}})$
$+ 4(n.|E(G_{-2m})| + m.|E(G_{2n})|) + 2mC_2.2nC_2 + 2n.2mC_2 + 2m.2nC_2$

+ number of $P_3$s in $-G_{2m}$ such that each $P_3$ is not a subgraph of any $C_3$ of $-G_{2m}$

+ number of $P_3$s in $G_{2n}$ such that each $P_3$ is not a subgraph of any $C_3$ of $G_{2n}$.

$= \frac{(m-1)m(7m^2-31m+34)}{24} + \frac{(n-1)n(7n^2-31n+34)}{24} + 3(\frac{(m-2)(m-1)m}{3} + \frac{(n-2)(n-1)n}{3}) + 4mn(m-1)$
$+ 4mn(n-1) + mn(2m-1)(2n-1) + 2mn(2m-1) + 2mn(2n-1) + \frac{(m-1)m(2m-1)}{6} + \frac{(n-1)n(2n-1)}{6}$

$= \frac{(m-1)m(7m^2+m-18)}{24} + \frac{(n-1)n(7n^2+n-18)}{24} + mn(4mn + 6(m+n) - 11)$.

(ii) $|C_4|_{G_{-2m,2n+1}} = |C_4|_{G_{2m}} + |C_4|_{G_{2n+1}} + 3(|C_3|_{G_{2m}} + |C_3|_{G_{2n+1}}) + 2((2n+1)|E(-G_{2m})|$
$+2m.|E(G_{2n+1})|) + 2mC_2.(2n+1)C_2 + (2n+1).2mC_2 + 2m.(2n+1)C_2$
+ number of $P_3$s in $-G_{2m}$ such that each $P_3$ is not a subgraph of any $C_3$ of $-G_{2m}$
+ number of $P_3$s in $G_{2n+1}$ such that each $P_3$ is not a subgraph of any $C_3$ of $G_{2n+1}$

$= \frac{(m-2)(m-1)m(7m-17)}{24} + \frac{(n-2)(n-1)n(7n-1)}{24} + (m-2)(m-1)m + \frac{(n-1)n(2n-1)}{2}$

$+ 2(2n+1)(m-1)m + 4mn^2 + m(2m-1)(2n+1)n + (2n+1)m(2m-1)$

$+ 2m(2n+1)n + \frac{(m-1)m(2m-1)}{6} + \frac{(n-1)n(n+1)}{3}$

$= \frac{(m-1)m(7m^2+m-18)}{24} + \frac{(n-1)n(7n^2+17n-2)}{24} + m(4m-3)(2n+1) + mn(4mn+2m+6n+1)$.

Similarly, we obtain,

(iii) $|C_4|_{G_{-(2m+1),2n}} = \frac{(m-1)m(7m^2+17m-2)}{24} + \frac{(n-1)n(7n^2+n-18)}{24}$
$+ (2m+1)n(4n-3) + mn(4mn+6m+2n+1)$.

(iv) $|C_4|_{G_{-(2m+1),2n+1}} = |C_4|_{G_{2m+1}} + |C_4|_{G_{2n+1}} + 3(|C_3|_{G_{2m+1}} + |C_3|_{G_{2n+1}})$
$+ 2((2n+1).|E(-G_{2m+1})| + (2m+1).|E(G_{2n+1})|)$

$+ (2m+1)C_2.(2n+1)C_2 + (2n+1).(2m+1)C_2 + (2m+1).(2n+1)C_2$

+ number of $P_3$s in $-G_{(2m+1)}$ such that each $P_3$ is not a subgraph of any $C_3$ of $-G_{(2m+1)}$

+ number of $P_3$s in $G_{2n+1}$ such that each $P_3$ is not a subgraph of any $C_3$ of $G_{2n+1}$

$= \frac{(m-2)(m-1)m(7m-1)}{24} + \frac{(n-2)(n-1)n(7n-1)}{24} + \frac{(m-1)m(2m-1)}{2} + \frac{(n-1)n(2n-1)}{2}$
$+ 2(2n+1)m^2 + 2(2m+1)n^2 + (2m+1)mn(2n+1)$

$+ (2n+1)(2m+1)m + (2m+1)(2n+1)n + \frac{(m-1)m(m+1)}{3} + \frac{(n-1)n(n+1)}{3}$

$= \frac{(m-1)m(7m^2+17m-2)}{24} + \frac{(n-1)n(7n^2+17n-2)}{24}$
$+ 2(m^2(2n+1) + (2m+1)n^2) + (mn+m+n)(2m+1)(2n+1)$.

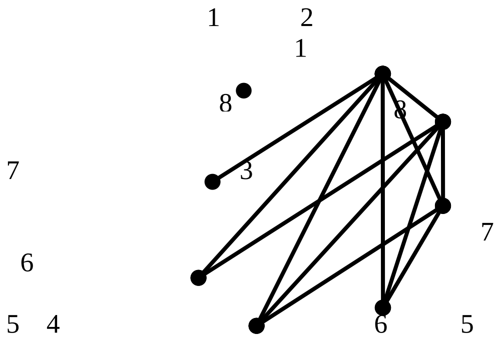
Results (v) – (viii) follow from $G^c_{-m,n} = K_1(0) \cup (-G^c_m) \cup G^c_n$ and using Theorem 2.2.□

Any property of natural numbers is interesting and important. From Theorems 2.1, 2.2, 2.6 and Corollary 2.9, we obtain the following simple properties of natural numbers.

**Theorem 2.10** *For* $2 \le n$, $n(n+1)(7n-4)$, $n(n+1)(7n+8)$ and $n(7n^2+18n+5)$ are divisible by 6 and $n(n+1)(n+2)(7n-3)$, $n(n+1)(n+2)(7n+1)$, $n(n+1)(n+2)(7n+13)$, $n(n+1)(7n^2+15n-10)$ and $n(n+1)(7n^2+31n+22)$ are divisible by 24, $m,n\in N$.

**Proof:** Result follows from Theorems 2.1,2.1,2.9, 2.1, 2.1, 2.1, 2.9 and 2.9, respectively.□

**Acknowledgement** I express my sincere thanks to Prof. Lowel W. Beineke, Indiana-Purdue University, USA, Prof. V. Mohan, Thiyagarajar College of Engineering, Madurai, India, Prof. M.I. Jinnah, University of Kerala, Thiruvananthapuram, India andDr. A. P. Santhakumaran, Professor, Department of Mathematics, Hindustan University, Chennai, India for their valuable help and guidance.



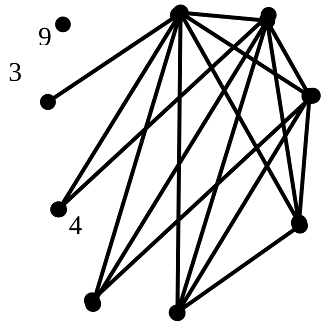




Fig. 1.$G_8$.Fig. 2. $G_9$.

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