International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 4, April 2014, PP 330-344
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Some Elementary Problems from the Note Books of Srinivasa Ramanujan V

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Entry 29 (p 369) Page80 of Berndt Vol IV

Problem 54: **NBSR Vol II p 369**

Let $1,\alpha,\alpha^2,\alpha^3,\alpha^4,\alpha^5,\alpha^6$ be the seventh roots of 1. Then

$$\left(\alpha+\alpha^2+\alpha^4\ ;\ \alpha^3+\alpha^5+\alpha^6\right)=\frac{1}{2}\left(-1\pm i\sqrt{7}\right)$$

Proof: The seventh roots of 1 ($\alpha^k, k=0,1,2,3,4,5,6$) satisfy the equation

$$z^7=1 \quad i.e.,\ z^7-1=0 \tag{1}$$

$\therefore$ Sum of the roots $=1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=(1-\alpha^7)(1-\alpha)=0$ (2)

and the product of the roots $=1.\alpha.\alpha^2.\alpha^3.\alpha^4.\alpha^5.\alpha^6=\alpha^{1+2+3+4+5+6}=\alpha^{21}=(\alpha^7)^3=1$ (3)

The sum of the given two expressions $(\alpha+\alpha^2+\alpha^4)$ *and* $(\alpha^3+\alpha^5+\alpha^6)$

$$=\alpha+\alpha^2+\alpha^4+\alpha^3+\alpha^5+\alpha^6=-1 \quad from\ eq.(2)$$

and their product $=(\alpha+\alpha^2+\alpha^4)(\alpha^3+\alpha^5+\alpha^6)$

$$=\alpha^4(1+\alpha+\alpha^3)(1+\alpha^2+\alpha^3)$$

$$=\alpha^4(1+\alpha+\alpha^2+3\alpha^3+\alpha^4+\alpha^5+\alpha^6)$$

$$=\alpha^4\{(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6)+2\alpha^3\}$$

$=\alpha^4(0+2\alpha^3)=2\alpha^7=2$ (since α is a seventh root of 1)

$\therefore$ the given two expressions are the roots of the quadratic equation: $x^2+x+2=0$ (4)

The roots of the quadratic (4) are

$$\frac{-1\pm\sqrt{1-4\times2}}{2}=\frac{1}{2}\left(-1\pm i\sqrt{7}\right) \tag{5}$$

$\therefore \quad (1+\alpha^2+\alpha^4);\ (\alpha^3+\alpha^5+\alpha^6)=\frac{1}{2}\left(-1\pm i\sqrt{7}\right)$

Note: The next two entries are rather difficult at the lower level. However a knowledge of the series expansions for Sin x and Cos x and sum of a Geometric Progression are needed in establishing the entries 30 and 31 jotted by S.R on p 369 of NBSR.

$$Sin\,x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ \ldots\ldots\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

and

$$Cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots\ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

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 # Entry 31 (p 369)

Problem 55: **NBSR Vol II p 369**

Let $\emptyset, \Psi$ and α be as defined in the previous two jottings. Then

$$64\prod_{k=0}^{6} Sin\,(\alpha^k x) = -\emptyset(2x) - \emptyset\{2(\alpha+\alpha^2+\alpha^4)x\} - \emptyset\{2(\alpha^3+\alpha^5+\alpha^6)x\}$$
$$+\emptyset\{2(\alpha+\alpha^6)x\} + \emptyset\{2(\alpha^2+\alpha^5)x\} + \emptyset\{2(\alpha^3+\alpha^4)x\}$$
$$+\emptyset\left\{\frac{2x}{\alpha+\alpha^6}\right\} + \emptyset\left\{\frac{2x}{\alpha^2+\alpha^5}\right\} + \emptyset\left\{\frac{2x}{\alpha^3+\alpha^4}\right\} \qquad (1)$$

and

$$64\prod_{k=0}^{6} Cos\,(\alpha^k x) = \Psi(2x) + \Psi\{2(\alpha+\alpha^2+\alpha^4)x\} - \Psi\{2(\alpha^3+\alpha^5+\alpha^6)x\}$$
$$+\Psi\{2(\alpha+\alpha^6)x\} + \Psi\{2(\alpha^2+\alpha^5)x\} + \Psi\{2(\alpha^3+\alpha^4)x\}$$
$$+\Psi\left\{\frac{2x}{\alpha+\alpha^6}\right\} + \Psi\left\{\frac{2x}{\alpha^2+\alpha^5}\right\} + \Psi\left\{\frac{2x}{\alpha^3+\alpha^4}\right\}$$

The above two results are beyond the comprehension of the readers of the present book.

A remark by Ramanujan:

.........“ from which 7 interval formula can be found ”.

Notes: A “7 interval formula” is a recursive formula for Bernoulli numbers wherein the differences of successive indices are 14. Ramanujan does not state this “7 interval formula”.

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Problem 56:

Prove that $16\, Cos\, 20\, Cos\, 40\, Cos\, 60\, Cos\, 80 = 1$ *

Solution: $16\, Cos\, 20\, Cos\, 40\, Cos\, 60\, Cos\, 80$

$$= 4\{2\, Cos\, 20 Cos\, 80\}\{2 Cos\, 40\, Cos\, 60\}$$
$$= 4\{Cos\, 100 + Cos\, 60\}\{Cos\, 100 + Cos\, 20\}$$
$$= 4\{Cos\, 60 - Cos\, 80\}\{Cos\, 20 - Cos\, 80\}$$
$$= 4\left\{\frac{1}{2} Cos\, 20 - \frac{1}{2} Cos\, 80 - Cos\, 80\, Cos\, 20 + 2 Cos^2 80\right\}$$
$$= 2[\{Cos\, 20 - Cos\, 80\} - \{Cos\, 100 + Cos\, 60\} + \{1 + Cos 160\}]$$
$$= 2\left\{Cos\, 20 - Cos\, 80 + Cos\, 80 - \frac{1}{2} + 1 - Cos\, 20\right\}$$
$$= 2\left\{\frac{1}{2} - Cos\, 20 + Cos\, 20\right\} = 1$$

Hence the result *.

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Scribbling of S.R (NBSR Vol II p 356)

Problem 57 :

Show that

$$\sqrt[3]{Cos\ 40^0} + \sqrt[3]{Cos\ 80^0} = \sqrt[3]{Cos\ 20^0} + \sqrt[3]{\frac{3}{2}\left(\sqrt[3]{9} - 2\right)} \quad (*.1)$$

and $$\sqrt[3]{Sec\ 40^0} + \sqrt[3]{Sec\ 80^0} = \sqrt[3]{Sec20^0} + \sqrt[3]{6\left(\sqrt[3]{9} - 1\right)} \quad (*.2)$$

Solution:

Note: The superscript (^{0}) of the angles indicate that the angles are given in degree measure. The superscript (^{0}) is dropped in the following steps.

Consider the trigonometric identity:

$$Cos\ 3\theta = 4Cos^3\theta - 3Cos\ \theta$$

$$\Rightarrow (2Cos\ \theta)^3 = 3(2\ Cos\ \theta) + 2\ Cos\ 3\theta \quad (1)$$

Now $$(2Cos\ 20)^3 = 3(2\ Cos\ 20) + 2\ (Cos\ 60) = 3(2\ Cos\ 20) + 1 \quad (2.1)$$

$$(2Cos\ 40)^3 = 3(2\ Cos\ 40) + 2\ (Cos\ 120) = 3(2\ Cos\ 40) - 1$$

$$\Rightarrow (-2Cos\ 40)^3 = 3(-2\ Cos\ 40) + 1 \quad (2.2)$$

and also

$$(2Cos\ 80)^3 = 3(2\ Cos\ 80) + 2\ (Cos\ 240) = 3(2\ Cos\ 80) - 1$$

$$\Rightarrow (-2Cos\ 80)^3 = 3(-2\ Cos\ 80) + 1 \quad (2.3)$$

The results (2.1) – (2.3) suggest that $2Cos\ 20\ ; -2\ Cos\ 40\ ; -2\ Cos\ 80$ are roots of the cubic equation:

$$z^3 - 3z - 1 = 0 \quad (3)$$

Let us refer to the problem: (NBSR Vol II p 325)

If α, β, γ be the roots of the equation $x^3 - ax^2 + bx - 1 = 0$

Then $$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{a + b + 3t}$$

and

$$\sqrt[3]{\alpha\beta} + \sqrt[3]{\beta\gamma} + \sqrt[3]{\gamma\alpha} = \sqrt[3]{b + 6 + 3t}$$

where t satisfying the cubic equation $t^3 - 3(a + b + 3)t - \{ab + 6(a + b) + 9\} = 0$

In this put $a = 0, b = -3$

Then α, β, γ are the roots of the equation $x^3 - 3x - 1 = 0$ which is same as equation (3)

$\alpha = 2Cos\ 20\ ;\quad \beta = -2Cos40 \quad and \quad \gamma = -2Cos80$ are the roots of the equation (1).

Further, t is given by $t^3 + 9 = 0 \Rightarrow t = \sqrt[3]{-9} = -\sqrt[3]{9}$

$$(Cos\ 20)^{1/3} + (-2Cos\ 40)^{1/3} + (-2\ Cos\ 80)^{1/3} = (6 + 3t)^{1/3} = \left\{6 + 3\left(-\sqrt[3]{9}\right)\right\}^{1/3}$$

$$(Cos\ 20)^{1/3} - (Cos\ 40)^{1/3} - (\ Cos\ 80)^{1/3} = \left\{\frac{3}{2}\left(2 - \sqrt[3]{9}\right)\right\}^{1/3} = -\left\{\frac{3}{2}\left(2 - \sqrt[3]{9}\right)\right\}^{1/3}$$

$$(Cos\ 40)^{1/3} + (Cos\ 80)^{1/3} = (\ Cos\ 20)^{1/3} + \left\{\frac{3}{2}\left(2 - \sqrt[3]{9}\right)\right\}^{1/3} \quad (4)$$

Similarly

$$\{(2Cos\,20)(-2Cos\,40)\}^{1/3}+\{(-2Cos\,40)(-2Cos\,80)\}^{1/3}+\{(-2Cos\,80)(2Cos\,20)\}^{1/3}$$
$$=\{3+3t\}^{1/3}$$

$$\{-4\,Cos\,20\,Cos\,40\}^{1/3}+\{4\,Cos\,40\,Cos\,80\}^{1/3}+\{-4\,Cos\,80\,Cos\,20\}^{1/3}=\{3(1-\sqrt[3]{9})\}^{1/3} \quad (5)$$

But $8\,Cos\,20\,Cos\,40\,Cos\,80=1$

$\therefore\ 4\cos 20\,Cos\,40=\frac{Sec\,80}{2}$; $4\,Cos\,40\,Cos\,80=\frac{Sec\,20}{2}$ and $4\,Cos\,80Cos\,20=\frac{Sec\,40}{2}$

Substituting these values in the result (5) and rearranging we get

$$\left\{\frac{1}{2}\,Sec\,40\right\}^{1/3}+\left\{\frac{1}{2}\,Sec\,80\right\}^{1/3}=\left\{\frac{1}{2}\,Sec\,20\right\}^{1/3}+\{3(\sqrt[3]{9}-1)\}^{1/3}$$

$i.e.,$ $$\{Sec\,40\}^{1/3}+\{Sec\,80\}^{1/3}=\{Sec\,20\}^{1/3}+\{6(\sqrt[3]{9}-1)\}^{1/3}$$

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Notation given on p 56 in Vol II NBSR

$I(p)$: the greatest integer $\leq p$) i.e., $I(p)=$ integral part of p

Examples: $I(7)=7$; $I\left(\frac{22}{3}\right)=7$ and so on. $I(p)$ is also denoted by $[p]$ or $\lfloor p \rfloor$.

$N(p)$: the integer nearest to p.

Examples: $N(7)=7; N\left(\frac{32}{5}=6\frac{2}{5}\right)=6; N\left(\frac{34}{5}=6\frac{4}{5}\right)=7;$

$N\left(\frac{50}{7}=7\frac{1}{7}\right)=7; N\left(\frac{55}{7}=7\frac{6}{7}\right)=8$ and so on.

Note: This definition for $N(p)$ of S.R is ambiguous when $p+1/2$ is an integer. In such a situation, it is obvious from the sequel that $N(p)=p+\frac{1}{2}$ instead of $p-\frac{1}{2}$.

$G(p)$: the least integer $\geq p$

$G(7)=7$; $G\left(\frac{50}{7}=7\frac{1}{7}\right)=8$ and so on.

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Scribbling of S.R

$N(p)=I\left(p+\frac{1}{2}\right)$ *

Problem 58: **NBSR Vol II p 267**

For each real number p

$$N(p)=I\left(p+\frac{1}{2}\right)$$

Solution:

Let $p=[p]+\{p\}$ (1)

where $\{p\}$ is the fractional part of p. Evidently $0<\{p\}<1$

If $\{p\}<\frac{1}{2}$, then $N(p)=[p]$ and $I\left(p+\frac{1}{2}\right)=[p]$.

Further, if $\{p\}>\frac{1}{2}$, then $N(p)=[p]+1$ and $I\left(p+\frac{1}{2}\right)=[p]+1$

In either case, $N(p) = I\left(p + \frac{1}{2}\right)$

The result * is established for

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Scribbling of S.R

$I\left(\frac{n}{p}\right)$ is the coefficient of x^n in the expansion of $\frac{x^p}{(1-x)(1-x^p)}$

Problem 59: **NBSR Vol II p 267**

If p and n are positive integers, then the coefficient of x^n in the series expansion (in positive powers of $\frac{x^p}{(1-x)(1-x^p)}$ is $I\left(\frac{n}{p}\right)$

Solution:

$$\frac{x^p}{(1-x)(1-x^p)} = x^p(1-x)^{-1}(1-x^p)^{-1}$$

$$= x^p\{1 + x + x^2 + x^3 + \ldots\ldots + x^i + \ldots\ldots\}$$

$$\times\{1 + x^p + x^{2p} + x^{3p} + x^{4p} + \ldots\ldots\ldots + x^{jp} + \ldots\ldots\}$$

$$= \{1 + x + x^2 + x^3 + \ldots\ldots + x^i + \ldots\ldots\}$$

$$\times\{x^p + x^{2p} + x^{3p} + x^{4p} + \ldots\ldots\ldots + x^{(j+1)p} + \ldots\ldots\} \qquad (1)$$

Estimation of the coefficient of x^n in the above product (1):

The coefficients (of powers of x) in both the factors in (1) are the same and each is equal to 1.

$\therefore$ coefficient of x^n in the product (1) i.e., the series expansion of the L.H.S of (1) is equal to the number of times x^n gets repeated by the term by term multiplication of the two (factor) series of the product (1).

The indices of the terms in the first factor are all equal to one and those of the terms in the second factor are multiples of p $(i.e. p, 2p, 3p, 4p \ldots\ldots)$

The terms with x^n in the product (1) would be got by the multiplication of term pairs contained in the set:

$$\{(1,x^n);\ (x,x^{n-1});\ (x^2,x^{n-2})\ldots\ldots\ldots\ (x^k,x^{n-k})\ldots\ldots\ldots\ (x^{n-p},x^p)\}$$

$$= \left[\left\{x^k,(x^p)^{\frac{n-k}{p}}\right\},\qquad k = 0,1,\ldots\ldots..n-p\right]$$

The index of x^p in the second member of a typical pair in the set must be an integer.

$\therefore$ The number of such acceptable pairs $= I\left(\frac{n}{p}\right)$

$\therefore$ The coefficient of x^n in the expansion of $\frac{x^p}{(1-x)(1-x^p)}$ is $I\left(\frac{n}{p}\right)$.

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Examples:

1. The coefficient of x^{10} in the expansion $\frac{x^3}{(1-x)(1-x^3)}$ is $I\left(\frac{10}{3}\right) = 3$

$$\frac{x^3}{(1-x)(1-x^3)} = x^3\{1+x+x^2+x^3+\ \ldots\ldots\}\{1+x^3+x^6+x^9+\ \ldots\ldots\}$$

$$= \{1+x+x^2+x^3+\ \ldots\ldots\}\{x^3+x^6+x^9+x^{12}+\ \ldots\ldots\}$$

Terms with x^{10} can be got by multiplying the term pairs $(x,\ x^9)$; (x^4, x^6) and (x^7, x^3) (no. of pairs = 3)

$\therefore$ The coefficient of $x^{10} = 3 = I\left(\frac{10}{3}\right)$

2. The coefficient of x^{20} in the expansion of $\frac{x^5}{(1-x)(1-x^3)}$

$$\frac{x^5}{(1-x)(1-x^3)} = x^2.\frac{x^3}{(1-x)(1-x^3)}$$

$\therefore$ The coefficient of x^{20} in the expansion $\frac{x^5}{(1-x)(1-x^3)}$

$= \text{coefficient of } x^{18} \text{ in the expansion of} \frac{x^3}{(1-x)(1-x^3)} = I\left(\frac{18}{3}\right) = 6$

Corollary:

The coefficient of x^n in the expansion of $\frac{x^q}{(1-x)(1-x^p)}$:

$$\frac{x^q}{(1-x)(1-x^p)} = x^{q-p}.\frac{x^p}{(1-x)(1-x^p)}$$

$\therefore$ The coefficient of x^n in the expansion $\frac{x^q}{(1-x)(1-x^p)}$

$= \text{coefficient of } \{x^{n-(q-p)}\} \text{ in the expansion of } \frac{x^p}{(1-x)(1-x^p)}$

$= \left[\frac{n-(q-p)}{p}\right] = \frac{n+p-q}{p}$

Note: The coefficient is zero if $n \leq (q-p)$

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Scribbling of S.R

The coefficient of x^{100} *in* $\frac{x^7}{(1-x^2)(1-x^3)}$

$= \text{coefficient of } x^{95} \text{ in } \frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}$

$= I\left(\frac{95}{2}\right) - I\left(\frac{95}{3}\right) = 16$ *

Problem 60: **NBSR Vol II p 361**

The coefficients of x^{100} and x^{95} in the power series expansions of

$\frac{x^7}{(1-x^2)(1-x^3)}$ *and* $\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}$ respectively are each equal to

$$\left[\frac{95}{2}\right] - \left[\frac{95}{3}\right] = 16$$

Solution: It can be noted that

$$\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = \frac{x^2(1-x^3)-x^3(1-x^2)}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2-x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2(1-x)}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2}{(1-x^2)(1-x^3)}$$

$$\therefore \text{ coefficient of } x^{100} \ in \ \frac{x^7}{(1-x^2)(1-x^3)} = \text{coefficient of } x^{95} \text{ in } \left\{\frac{x^2}{(1-x^2)(1-x^3)}\right\}$$

$$= \text{coefficient of } x^{95} \text{ in } \left\{\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}\right\}$$

$$= \text{coefficient of } x^{95} \text{ in } \left\{\frac{x^2}{(1-x)(1-x^2)}\right\} - \text{coefficient of } x^{95} \text{ in } -\frac{x^3}{(1-x)(1-x^3)}$$

$$= I\left(\frac{95}{2}\right) - I\left(\frac{95}{3}\right) = 47 - 31 = 16$$

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Scribbling of S.R:

$$I\left(\frac{n+4}{6}\right) - I\left(\frac{n-3}{6}\right) + I\left(\frac{n+2}{6}\right) = I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right) \qquad *$$

Problem 61: **NBSR Vol II p.361**

For each positive integral value for n , $I\left(\frac{n+4}{6}\right) - I\left(\frac{n-3}{6}\right) + I\left(\frac{n+2}{6}\right) = I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right)$

Solution: It can be noticed that

$$\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = \frac{x^2(1-x^3)-x^3(1-x^2)}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2-x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{x^2}{(1-x^2)(1-x^3)}$$

$$= \frac{x^2(1+x^3)}{(1-x^2)(1-x^6)}$$

$$= \frac{x^2(1-x+x^2)}{(1-x)(1-x^6)}$$

Thus $$\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = \frac{x^2}{(1-x)(1-x^6)} - \frac{x^3}{(1-x)(1-x^6)} + \frac{x^4}{(1-x)(1-x^6)} \qquad (1)$$

The coefficient of x^n in the expansion of $L.H.S$ of $* = I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right)$ (2)

The coefficient of x^n in the expansion of $R.H.S$ of $*$

$$= I\left(\frac{n+6-2}{6}\right) - I\left(\frac{n+6-3}{6}\right) + I\left(\frac{n+6-4}{6}\right)$$

$$= I\left(\frac{n+4}{6}\right) - I\left(\frac{n+3}{6}\right) + I\left(\frac{n+2}{6}\right) \qquad (3)$$

Hence equating (2)and (30 we get the result

$$I\left(\frac{n+4}{6}\right) - I\left(\frac{n+3}{6}\right) + I\left(\frac{n+2}{6}\right) = I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right)$$

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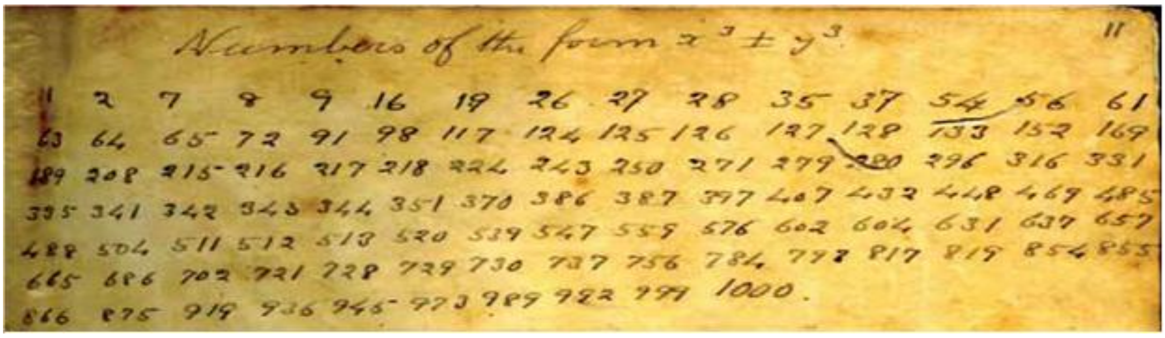
Problem 62: **SR Ms(3) p.11 and NBSR vol p371.**


Numbers of the form $x^3 \pm y^3$ 11

1 2 7 8 9 16 19 26 27 28 35 37 54 56 61
63 64 65 72 91 98 117 124 125 126 127 128 133 152 169
189 208 215 216 217 218 224 243 250 271 279 280 296 316 331
335 341 342 343 344 351 370 386 387 397 407 432 448 469 485
488 504 511 512 513 520 539 547 559 576 602 604 631 637 657
665 686 702 721 728 729 730 737 756 784 793 817 819 854 855
866 875 919 936 945 973 989 992 999 1000.

Comments: S.R in this scribbling lists out numbers from 1 to 1000 that can be expressed in the form $x^3 \pm y^3$ i. e., sum or difference between two cube numbers. By numbers he means positive integers only.

Separation of the numbers in to the form $x^3 \pm y^3$ can be seen from the following table.

$1 = 1^3 + 0^3$	$117 = 5^3 - 2^3$	$279 = 7^3 - 4^3$	$504 = 8^3 - 2^3$	**$728 = 12^3 - 10^3$**
$2 = 1^3 + 1^3$	$124 = 5^3 - 1^3$	$280 = 6^3 + 4^3$	$511 = 8^3 - 1^3$	$729 = 9^3 + 0^3$
$7 = 2^3 - 1^3$	$125 = 5^3 + 0^3$	$296 = 8^3 - 6^3$	$512 = 8^3 + 0^3$	$730 = 9^3 + 1^3$
$8 = 2^3 + 0^3$	$126 = 5^3 + 1^3$	$316 = 7^3 - 3^3$	**$513 = 8^3 + 1^3$**	$737 = 9^3 + 2^3$
$9 = 2^3 + 1^3$	$127 = 7^3 - 6^3$	$331 = 11^3 - 10^3$	**$513 = 9^3 - 6^3$**	$756 = 9^3 + 3^3$
$16 = 2^3 + 2^3$	$128 = 4^3 + 4^3$	$335 = 7^3 - 2^3$	$520 = 8^3 + 2^3$	$784 = 10^3 - 6^3$
$19 = 3^3 - 2^3$	$133 = 5^3 + 2^3$	$341 = 6^3 + 5^3$	$539 = 8^3 + 3^3$	$793 = 9^3 + 4^3$
$26 = 3^3 - 1^3$	**$152 = 5^3 + 3^3$**	$342 = 7^3 - 1^3$	$547 = 14^3 - 13^3$	$817 = 17^3 - 16^3$
$27 = 3^3 + 0^3$	**$152 = 6^3 - 4^3$**	$343 = 7^3 + 0^3$	$559 = 7^3 + 6^3$	$819 = 11^3 - 8^3$
$28 = 3^3 + 1^3$	$169 = 8^3 - 7^3$	$344 = 7^3 + 1^3$	$576 = 8^3 + 4^3$	$854 = 9^3 + 5^3$
$35 = 3^3 + 2^3$	**$189 = 5^3 + 4^3$**	$351 = 7^3 + 2^3$	$602 = 11^3 - 9^3$	$855 = 8^3 + 7^3$
$37 = 4^3 - 3^3$	**$189 = 6^3 - 3^3$**	$370 = 7^3 + 3^3$	$604 = 9^3 - 5^3$	$866 = 13^3 - 11^3$
$54 = 3^3 + 3^3$	$208 = 6^3 - 2^3$	$386 = 9^3 - 7^3$	$631 = 15^3 - 14^3$	$875 = 10^3 - 5^3$
$56 = 4^3 - 2^3$	$215 = 6^3 - 1^3$	$387 = 8^3 - 5^3$	$637 = 8^3 + 5^3$	$919 = 18^3 - 17^3$
$61 = 5^3 - 4^3$	$216 = 6^3 + 0^3$	$397 = 12^3 - 11^3$	$657 = 10^3 - 7^3$	$936 = 10^3 - 4^3$
$63 = 4^3 - 1^3$	**$217 = 6^3 + 1^3$**	$407 = 7^3 + 4^3$	$665 = 9^3 - 4^3$	$945 = 9^3 + 6^3$
$64 = 4^3 + 0^3$	**$217 = 9^3 - 8^3$**	$432 = 6^3 + 6^3$	$686 = 7^3 + 7^3$	$973 = 10^3 - 3^3$
$65 = 4^3 + 1^3$	$218 = 7^3 - 5^3$	$448 = 8^3 - 4^3$	$702 = 9^3 - 3^3$	**$988 = 11^3 - 7^3$ ***
$72 = 4^3 + 2^3$	$224 = 6^3 + 2^3$	**$468 = 7^3 + 5^3$ ***	**$721 = 9^3 - 2^3$**	$992 = 10^3 - 2^3$
$91 = 4^3 + 3^3$	$243 = 6^3 + 3^3$	$469 = 13^3 - 12^3$	**$721 = 16^3 - 15^3$**	**$999 = 10^3 - 1^3$**

$91 = 6^3 - 5^3$	$250 = 5^3 + 5^3$	$485 = 8^3 - 3^3$	**$728 = 8^3 + 6^3$**	**$999 = 12^3 - 9^3$**
$98 = 5^3 - 3^3$	$271 = 10^3 - 9^3$	$488 = 10^3 - 8^3$	**$728 = 9^3 - 1^3$**	$1000 = 10^3 + 0^3$

- There are two numbers missing in the entry (shown with * and bolded in the above table): **$468=5^3 + 7^3$** and **$988 = 11^3 - 7^3$.**
- 989 scribbled in the entry is expressible neither in the form $x^3 + y^3$ nor in the form $x^3 - y^3$. This might be a slip of S.R while scribbling 989 instead of 988.
- The table is extended further, scanning all numbers up to 5000 that can be expressed in the form $x^3 \pm y^3$.

Numbers between 1001 to 2000.

$1001 = 10^3 + 1^3$	**$1216 = 10^3 + 6^3$**	$1385 = 12^3 - 7^3$	$1647 = 15^3 - 12^3$	$1744 = 14^3 - 10^3$
$1008 = 10^3 + 2^3$	**$1216 = 12^3 - 8^3$**	$1387 = 22^3 - 21^3$	$1657 = 24^3 - 23^3$	$1755 = 12^3 + 3^3$
$1016 = 14^3 - 12^3$	$1241 = 9^3 + 8^3$	$1395 = 11^3 + 4^3$	$1664 = 12^3 - 4^3$	$1792 = 12^3 + 4^3$
$1024 = 8^3 + 8^3$	$1261 = 21^3 - 20^3$	$1413 = 14^3 - 11^3$	$1674 = 11^3 + 7^3$	$1801 = 25^3 - 24^3$
$1027 = 10^3 + 3^3$	$1267 = 11^3 - 4^3$	$1456 = 11^3 + 5^3$	$1685 = 13^3 - 8^3$	$1843 = 11^3 + 8^3$
$1027 = 19^3 - 18^3$	$1304 = 11^3 - 3^3$	$1458 = 9^3 + 9^3$	$1701 = 12^3 - 3^3$	$1853 = 12^3 + 5^3$
$1064 = 10^3 + 4^3$	$1323 = 11^3 - 2^3$	$1468 = 13^3 - 9^3$	$1720 = 12^3 - 2^3$	$1854 = 13^3 - 7^3$
$1072 = 9^3 + 7^3$	$1330 = 11^3 - 1^3$	**$1512 = 10^3 + 8^3$**	$1727 = 12^3 - 1^3$	$1899 = 16^3 - 13^3$
$1115 = 11^3 - 6^3$	$1331 = 11^3 + 0^3$	**$1512 = 12^3 - 6^3$**	$1728 = 12^3 + 0^3$	$1944 = 12^3 + 6^3$
$1125 = 10^3 + 5^3$	$1332 = 11^3 + 1^3$	$1519 = 23^3 - 22^3$	**$1729 = 10^3 + 9^3$**	$1946 = 19^3 - 17^3$
$1141 = 20^3 - 19^3$	$1339 = 11^3 + 2^3$	$1538 = 17^3 - 15^3$	**$1729 = 12^3 + 1^3$**	$1951 = 26^3 - 25^3$
$1178 = 15^3 - 13^3$	$1343 = 10^3 + 7^3$	$1547 = 11^3 + 6^3$	**$1736 = 12^3 + 2^3$**	$1981 = 13^3 - 6^3$
$1197 = 13^3 - 10^3$	$1352 = 16^3 - 14^3$	$1603 = 12^3 - 5^3$	**$1736 = 18^3 - 16^3$**	$2000 = 10^3 + 10^3$
$1206 = 11^3 - 5^3$	$1358 = 11^3 + 3^3$			

Numbers between 2001 to 3000.

$2015 = 14^3 - 9^3$	$2197 = 13^3 + 0^3$	$2401 = 14^3 - 7^3$	$2662 = 11^3 + 11^3$	$2765 = 16^3 - 11^3$
$2044 = 15^3 - 11^3$	$2198 = 13^3 + 1^3$	$2402 = 21^3 - 19^3$	$2680 = 14^3 - 4^3$	$2771 = 14^3 + 3^3$
$2060 = 11^3 + 9^3$	$2205 = 13^3 + 2^3$	$2413 = 13^3 + 6^3$	$2709 = 13^3 + 8^3$	$2791 = 31^3 - 30^3$
$2071 = 12^3 + 7^3$	$2224 = 13^3 + 3^3$	$2437 = 29^3 - 28^3$	$2716 = 17^3 - 13^3$	$2808 = 14^3 + 4^3$
$2072 = 13^3 - 5^3$	$2232 = 14^3 - 8^3$	**$2457 = 12^3 + 9^3$**	$2717 = 14^3 - 3^3$	$2863 = 15^3 - 8^3$
$2107 = 27^3 - 26^3$	$2240 = 12^3 + 8^3$	**$2457 = 18^3 - 15^3$**	$2728 = 12^3 + 10^3$	$2869 = 14^3 + 5^3$
$2133 = 13^3 - 4^3$	$2261 = 13^3 + 4^3$	$2528 = 14^3 - 6^3$	$2736 = 14^3 - 2^3$	$2906 = 23^3 - 21^3$

$2168 = 20^3 - 18^3$	$2269 = 28^3 - 27^3$	$2540 = 13^3 + 7^3$	$2743 = 14^3 - 1^3$	$2926 = 13^3 + 9^3$
$2169 = 17^3 - 14^3$	$2322 = 13^3 + 5^3$	$2611 = 30^3 - 29^3$	$2744 = 14^3 + 0^3$	$2960 = 14^3 + 6^3$
$2170 = 13^3 - 3^3$	$2331 = 11^3 + 10^3$	$2619 = 14^3 - 5^3$	$2745 = 14^3 + 1^3$	$2977 = 32^3 - 31^3$
$2189 = 13^3 - 2^3$	$2368 = 16^3 - 12^3$	$2646 = 15^3 - 9^3$	$2752 = 14^3 + 2^3$	
$2196 = 13^3 - 1^3$	$2375 = 15^3 - 10^3$	$2648 = 22^3 - 20^3$	$2763 = 19^3 - 16^3$	

Numbers between 3001 to 4000.

$3032 = 15^3 - 7^3$	$3197 = 13^3 + 10^3$	$3376 = 15^3 + 1^3$	$3528 = 13^3 + 11^3$	$3781 = 36^3 - 35^3$
$3059 = 12^3 + 11^3$	$3250 = 15^3 - 5^3$	$3383 = 15^3 + 2^3$	$3571 = 35^3 - 34^3$	$3789 = 22^3 - 19^3$
$3087 = 14^3 + 7^3$	$3256 = 14^3 + 8^3$	$3402 = 15^3 + 3^3$	$3582 = 17^3 - 11^3$	$3880 = 16^3 - 6^3$
$3087 = 20^3 - 17^3$	$3311 = 15^3 - 4^3$	$3429 = 21^3 - 18^3$	$3584 = 16^3 - 8^3$	$3887 = 15^3 + 8^3$
$3088 = 18^3 - 14^3$	$3348 = 15^3 - 3^3$	$3439 = 15^3 + 4^3$	$3591 = 15^3 + 6^3$	$3904 = 20^3 - 16^3$
$3096 = 16^3 - 10^3$	**$3367 = 15^3 - 2^3$**	$3456 = 12^3 + 12^3$	$3635 = 18^3 - 13^3$	$3913 = 17^3 - 10^3$
$3159 = 15^3 - 6^3$	**$3367 = 16^3 - 9^3$**	$3458 = 25^3 - 23^3$	$3718 = 15^3 + 7^3$	$3925 = 13^3 + 12^3$
$3169 = 33^3 - 32^3$	**$3367 = 34^3 - 33^3$**	$3473 = 14^3 + 9^3$	$3744 = 14^3 + 10^3$	$3971 = 16^3 - 5^3$
$3176 = 24^3 - 22^3$	$3374 = 15^3 - 1^3$	$3484 = 19^3 - 15^3$	$3752 = 26^3 - 24^3$	$3997 = 37^3 - 36^3$
$3185 = 17^3 - 12^3$	$3375 = 15^3 + 0^3$	$3500 = 15^3 + 5^3$	$3753 = 16^3 - 7^3$	

Numbers between 4001 to 5000.

$4032 = 16^3 - 4^3$	$4115 = 19^3 - 14^3$	$4394 = 13^3 + 13^3$	$4681 = 40^3 - 39^3$	$4905 = 17^3 - 2^3$
$4058 = 27^3 - 25^3$	$4123 = 16^3 + 3^3$	$4401 = 17^3 - 8^3$	$4697 = 17^3 - 6^3$	$4912 = 17^3 - 1^3$
$4069 = 16^3 - 3^3$	$4160 = 16^3 + 4^3$	$4439 = 16^3 + 7^3$	**$4706 = 15^3 + 11^3$**	$4913 = 17^3 + 0^3$
$4075 = 14^3 + 11^3$	$4167 = 23^3 - 20^3$	$4447 = 39^3 - 38^3$	**$4706 = 29^3 - 27^3$**	$4914 = 17^3 + 1^3$
$4088 = 16^3 - 2^3$	$4184 = 17^3 - 9^3$	$4472 = 14^3 + 12^3$	$4788 = 17^3 - 5^3$	**$4921 = 17^3 + 2^3$**
$4095 = 16^3 - 1^3$	$4219 = 38^3 - 37^3$	$4501 = 18^3 - 11^3$	$4816 = 22^3 - 18^3$	**$4921 = 41^3 - 40^3$**
$4096 = 16^3 + 0^3$	$4221 = 16^3 + 5^3$	$4563 = 24^3 - 21^3$	$4825 = 16^3 + 9^3$	$4940 = 17^3 + 3^3$
$4097 = 16^3 + 1^3$	$4312 = 16^3 + 6^3$	$4570 = 17^3 - 7^3$	$4832 = 18^3 - 10^3$	$4941 = 14^3 + 13^3$
$4104 = 15^3 + 9^3$	$4348 = 21^3 - 17^3$	$4608 = 16^3 + 8^3$	$4849 = 17^3 - 4^3$	**$4977 = 17 + 4^3$**
$4104 = 16^3 + 2^3$	$4375 = 15^3 + 10^3$	$4625 = 20^3 - 15^3$	$4886 = 17^3 - 3^3$	**$4977 = 25^3 - 22^3$**
$4104 = 18^3 - 12^3$	$4376 = 28^3 - 26^3$	$4662 = 19^3 - 13^3$		

	Analysis of the numbers of the form $x^3 \pm y^3$ between 1 to 5000				
Numbers of the form	Range of numbers				
	1-1000	1001 - 2000	2001 - 3000	3001-4000	4001 - 5000
$x^3 + y^3$ in more than one way. $N=x_1^3+y_1^3=x_2^3+y_2^3$ such numbers are popular as Ramanujan Numbers.		$1729 =10^3+9^3$ $=12^3+1^3$			$4104 = 15^3+9^3$ $= 16^3+2^3$
$x^3 - y^3$ in more than one way $N=x_1^3- y_1^3$ $= x_2^3- y_2^3$. These lead to Ramanujan Numbers: $x_1^3+y_2^3 = x_2^3+y_1^3$	$721= 9^3-2^3 =16^3-6^3$ $728= 9^3-1^3 =12^3-10^3$ $999=10^3-1^3=12^3-9^3$ → $2^3+16^3=9^3+15^3=4104$ $1^3+12^3=9^3+10^3=1729$			$3367=\begin{cases}15^3-2^3\\16^3-9^3\\34^3-33^3\end{cases}$ $4104 = \begin{cases}2^3+16^3\\9^3+15^3\end{cases}$ $39312=\begin{cases}2^3+34^3\\15^3+33^3\end{cases}$ $40033=\begin{cases}9^3+34^3\\16^3+33^3\end{cases}$	
both the forms $x^3 + y^3$ and $x^3 - y^3$ i. e $N= x_1^3+ y_1^3= x_2^3- y_2^3$ these lead to numbers of the form $x_1^3+y_1^3+y_2^3 = x_2^3$ (sum of the three cubes is a cube)	$91 =3^3+4^3 =6^3-5^3$ $152=3^3+5^3 =6^3-4^3$ $217=1^3+6^3 =9^3-8^3$ $189=4^3+5^3 =6^3-3^3$ $513=1^3+8^3 =9^3-6^3$ $728=6^3+8^3 =9^3-1^3$ → $3^3+4^3+5^3 = 6^3(216)$ $1^3+6^3+8^3 =9^3(=729)$	$1027 =3^3+10^3$ $=19^3-18^3$ $1216 =6^3+10^3$ $=12^3-8^3$ $1512=10^3+8^3$ $=12^3-6^3$ $1736=12^3+2^3$ $=18^3-16^3$ → $3^3+10^3+18^3=19^3$ (=6859) $6^3+8^3+10^3=12^3$ (=1728)	$2457=12^3+9^3$ $=18^3-15^3$ → $9^3+12^3+15^3=18^3$ (=5832)	$3087=7^3+14^3$ $=20^3-17^3$ → $7^3+14^3+17^3=20^3$ (=8000)	$4104=12^3 + 16^3$ $=18^3-12^3$ $4706=15^3 + 11^3$ $= 29^3 - 27^3$ $4921=17^3 + 2^3$ $= 41^3 + 40^3$ → $2^3+12^3+16^3=18^3$ (=5832)

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Problem 63 : **Lost NB SR p. 344**

$$\frac{(\sqrt{a^2+ab+b^2}-a)(\sqrt{a^2+ab+b^2}-b)}{a+b-\sqrt{a^2+ab+b^2}} = a+b \quad *$$

Solution:

$$\text{L.H. of } * = \frac{(a^2+ab+b^2)-(a+b)\sqrt{a^2+ab+b^2}+ab}{(a+b)-\sqrt{a^2+ab+b^2}}$$

$$= \frac{(a+b)^2 - (a+b)\sqrt{a^2+ab+b^2}}{(a+b) - \sqrt{a^2+ab+b^2}}$$

$$= \frac{(a+b)\ \ (a+b) - \sqrt{a^2+ab+b^2}}{((a+b) - \sqrt{a^2+ab+b^2})} \quad = \ (a+b)$$

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Problem 64: **Lost NB SR p. 344**

$$\sqrt[3]{(a+b)(a^2+b^2)} - a \quad \sqrt[3]{(a+b)(a^2+b^2)} - b$$

$$= \frac{\sqrt[3]{(a+b)^2} - \sqrt[3]{a^2+b^2}}{\sqrt[3]{(a+b)^2} + \sqrt[3]{a^2+b^2}}.(ab+ab+bc)$$

Solution:

L.H.S = $\sqrt[3]{(a+b)(a^2+b^2)} - a \quad \sqrt[3]{(a+b)(a^2+b^2)} - b$

$$= (a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^{\frac{1}{3}}(a^2+b^2)^{\frac{1}{3}}(a+b) + ab$$

$$= (a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{1}{3}} + ab$$

∴ (L.H.S) x Denominator of R.H.S

$$= \left\{(a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{1}{3}} + ab\right\}.\left\{(a+b)^{\frac{2}{3}}(a^2+b^2)^{\frac{1}{3}}\right\}$$

$$= (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{2}{3}} - (a+b)^2(a^2+b^2)^{\frac{1}{3}} + ab(a+b)^{\frac{2}{3}}$$

$$+ (a+b)^{\frac{2}{3}}(a^2+b^2) - (a+b)^{\frac{4}{3}}(a^2+b^2)^{\frac{1}{3}} + ab(a^2+b^2)^{\frac{1}{3}}$$

$$= (a^2+b^2)^{\frac{1}{3}} \ \ ab - (a+b)^2 \ \ + (a+b)^{\frac{2}{3}}(a^2+b^2+ab)$$

= Numerator of (R.H.S)

∴ L.H.S = R.H.S

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Problem 65: **Lost NB SR p. 313**

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\} = \frac{\left(1+\frac{1+2\beta}{n}\right)\left(1+\frac{2\alpha+\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3} x$$

$$\left\{1-\left[\frac{(\alpha-\beta)+i(\alpha+\beta)^{\sqrt{3}}}{2n}\right]^2\right\}.\left\{1-\left[\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right]^2\right\}$$

Solution:

The Standard identity $1 + x^3 = (1 + x)\ (1 - x^3)\ (1 - x^3 + x^2)$ will be employed in establishing the above identity.

Now $\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}=\left\{1+\frac{\alpha+\beta}{n+\alpha}\right\}\left\{1-\frac{\alpha+\beta}{n+\alpha}+\left(\frac{\alpha+\beta}{n+\alpha}\right)^2\right\}$

$$=\left\{\frac{n+2\alpha+\beta}{n+\alpha}\right\}\left\{\frac{(n+\alpha)^2-(\alpha+\beta)(n+\alpha)+(\alpha+\beta)^2}{(n+\alpha)^2}\right\}$$

$$=\frac{n}{(n+\alpha)^3}\left\{1+\frac{2\alpha+\beta}{n}\right\}\ n^2+n(\alpha-\beta)+\alpha^2+\alpha\beta+\beta^2$$

$$=\frac{1}{n^2(1+\frac{\alpha}{n})^3}\left\{1+\frac{2\alpha+\beta}{n}\right\}\ n^2+n(\alpha-\beta)+(\alpha^2+\alpha\beta+\beta^2)$$

The roots of the quadratic equation

$$n^2+n(\alpha-\beta)+(\alpha^2+\alpha\beta+\beta^2)=0$$

are n $=\frac{-(\alpha-\beta)\pm\sqrt{(\alpha-\beta)^2-4(\alpha^2+\alpha\beta+\beta^2)}}{2}$

$$=\frac{-(\alpha-\beta)\pm\sqrt{-3(\alpha^2+2\alpha\beta+\beta^2)}}{2}$$

$$=\frac{-(\alpha-\beta)\pm i(\alpha+\beta)\sqrt{3}}{2}$$ (= root 1, root 2 say)

$$\therefore\ n^2+n(\alpha-\beta)+(\alpha^2+\alpha\beta+\beta^2)=(n-root1)(n-root2)$$

$$=\left\{n-\frac{-(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2}\right\}\left\{n-\frac{-(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2}\right\}$$

$$=n^2\left\{1+\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}\left\{1+\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2}\right\}$$

We thus have

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}=\frac{1}{\left(1+\frac{\alpha}{n}\right)^3}\left\{1+\frac{2\alpha+\beta}{n}\right\}\left\{1+\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}\left\{1+\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}\quad -(1)$$

Similarly we get by interchanging α and β in the above result,

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}=\frac{1}{\left(1+\frac{\beta}{n}\right)^3}\left\{1+\frac{2\beta+\alpha}{n}\right\}.\left\{1+\frac{\beta-\alpha-i(\alpha+\beta)\sqrt{3}}{2n}\right\}\left\{1+\frac{(\beta-\alpha)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}\quad -(2)$$

Multiplying these two results **1** and **2** re-arranging the factors

We notice that $\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}=\frac{\left(1+\frac{2\alpha+\beta}{n}\right)\left(1+\frac{2\beta+\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}.$

$$.\left\{1+\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}.\left\{1+\frac{(\beta-\alpha)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}.$$

$$.\left\{1+\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}.\left\{1+\frac{(\beta-\alpha)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}$$

$$= \frac{\left(1+\frac{2\alpha+\beta}{n}\right)\left(1+\frac{2\beta+\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\left\{1-\left[\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right]^2\right\}\left\{1-\left[\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right]^2\right\}$$

The result is established.

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Problem 66 : **Lost NB SR p. 116**

Suppose that $\phi(x)=1+24\left\{\frac{x}{1-x}+\frac{3x^3}{1-x^3}+\frac{5x^5}{1-x^5}+....\right\}$, then $\phi(x)+\phi(-\sqrt{x})=2\phi(x)$

Solution:

$$\phi(x)=1+24\left\{\frac{x}{1-x}+\frac{3x^3}{1-x^3}+\frac{5x^5}{1-x^5}+....+\frac{(2n-1)x^{2n-1}}{1-x^{2n-1}}+...\right\} \quad \text{-(1)}$$

Now $(\sqrt{x})^{2n-1}=x^{(2n-1)/2}=\frac{x^n}{\sqrt{x}}=\sqrt{x}.x^{n-1}$ and $(-\sqrt{x})^{2n-1}=\sqrt{x}.x^{n-1}$ - (2)

$$\therefore\ \phi(\sqrt{x})=1+24\left\{\frac{\sqrt{x}}{1-\sqrt{x}}+\frac{3x\sqrt{x}}{1-x\sqrt{x}}+\frac{5.x^2\sqrt{x}}{1-x^2\sqrt{x}}+....+\frac{(2n-1)x^{n-1}\sqrt{x}}{1-x^{n-1}\sqrt{x}}+...\right\} \quad \text{-(3)}$$

and

$$\phi(-\sqrt{x})=1+24\left\{\frac{-\sqrt{x}}{1+\sqrt{x}}-\frac{3x\sqrt{x}}{1+x\sqrt{x}}-\frac{5.x^2\sqrt{x}}{1+x^2\sqrt{x}}-....-\frac{(2n-1)x^{n-1}\sqrt{x}}{1+x^{n-1}\sqrt{x}}+...\right\} \quad \text{-(4)}$$

To establish the result we add (3) and (4) and get:

$\emptyset(\sqrt{x})+\emptyset(-\sqrt{x})$

$$= 2+24\left[\sqrt{x}\left\{\frac{1}{1-\sqrt{x}}-\frac{1}{1+\sqrt{x}}\right\}+3x\sqrt{x}\left\{\left\{\frac{1}{1-x\sqrt{x}}-\frac{1}{1+x\sqrt{x}}\right\}+5x^2\sqrt{x}\left\{\frac{1}{1-x^2\sqrt{x}}-\frac{1}{1+x^2\sqrt{x}}\right\}\right\}+\right.$$

$$+\,.\,.\,.\,.\,.\,.\,.\,.\,.\,+$$

$$\left.(2n-1)x^{n-1}\sqrt{x}\left\{\left\{\frac{1}{1-x^{n-1}\sqrt{x}}-\frac{1}{1+x^{n-1}\sqrt{x}}\right\}\right\}+\,.\,.\,.\,.\,.\,.\right]$$

$$=2+24\left[\frac{\sqrt{x}.2\sqrt{x}}{1-x}+\frac{3.x\sqrt{x}.x\sqrt{x}}{1-x^3}+\frac{5x^2\sqrt{x}.x^2\sqrt{x}}{1-x^5}+...+\frac{(2n-1)x^{n-1}\sqrt{x}.x^{n-1}\sqrt{x}}{1-x^{2n-1}}+....\right]$$

$$=2\left[1+24\left\{\frac{x}{1-x}+\frac{3x^3}{1-x^3}+\frac{5x^5}{1-x^5}+.....+\frac{(2n-1)x^{2n-1}}{1-x^{2n-1}}+...\right\}\right]$$

$$=2\,\phi(\sqrt{x})$$

This establishes the result.

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----------------- (To be continued ……)

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