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On Orthogonal Double Covers of Complete Bipartite Graphs by Disjoint Unions of Graph-Paths

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Abstract: *We are concerned with the Cartesian products of two symmetric starter vectors of orthogonal double covers (ODCs) of the complete bipartite graphs and use this method to construct ODCs by new infinite classes of disjoint unions of graph-paths.*

Keywords: *Graph decomposition, Orthogonal double cover, Symmetric starter, Cartesian product, graph-paths.*

1. Introduction

Let X and G be graphs, such that G is isomorphic to a subgraph of X. An ODC of X by G is a collection $\mathfrak{B} =\{ f(x) : x \in V(x)\}$ of subgraphs of X, all isomorphic to G, such that (i) every edge of X occurs in exactly two members of $\mathfrak{B}$ and (ii) $f(x)$ and $f(y)$ share an edge if and only if x and y are adjacent in X. The elements of $\mathfrak{B}$ will be called pages. This concept is a natural generalization of earlier definitions of an ODC for complete and complete bipartite graphs, that have been studied extensively (see the survey [1]). An obvious necessary condition is that X is regular. An effective technique to construct ODCs in the above cases was based on the idea of translating a given subgraph of G by a group acting on $V(x)$. Here we deal with complete bipartite graph $K_{n,n}$ with partition sets of size n each.

In our paper, we use the usual notaion: $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_{m+1} for the path on m+1 vertices, C_n for the cycle on n vertices, $H \cup F$ for the disjoint union of H and F, and lG for l disjoint copies of G. In [2], the definitions not introduced here can be found.

The vertices of $K_{n,n}$ will be labeled by the elements of $\mathbb{Z}_n \times \mathbb{Z}_2$. Namely, for

$(v,i) \in \mathbb{Z}_n \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{a_i,b_j\} \in E(K_{n,n})$ if and only if $i \neq j$ for all $a,b \in \mathbb{Z}_n$ and $i,j \in \mathbb{Z}_2$. If there is no chance of confusion a_0b_1 will be written instead of $\{a_0,b_1\}$ for the edge between the vertices a_0,b_1.

Let G be a spanning subgraph of $K_{n,n}$ and let $x \in \mathbb{Z}_n$. Then the graph $G+x$ with $E(G+x) = \{(u+x,v+x) : (u,v) \in E(G)\}$ is called the *x-translate* of G. The length of an edge $e=(u,v)$ is defined by $d(e) = v-u$. Note that sums and differences are calculated modulo n.

G is called a *half starter* with respect to $\mathbb{Z}_n$ if $|E(G)| = n$ and the lengths of all edges in G are mutually distinct; that is, $\{d(e): e \in E(G)\} = \mathbb{Z}_n$. The following three results were proved in [3].

(i) If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$.

Hereafter, a half starter will be represented by the vector $v(G) = (v_0,v_1,\ldots,v_{n-1}) \in \mathbb{Z}_n^n$.

Two half starter vectors $v\,(G_0) = (v_0,v_1,\ldots,v_{n-1})$ and $v\,(G_1) = (u_0,u_1,\ldots,u_{n-1})$ are said to be orthogonal if $\{v_i - u_i : i \in \mathbb{Z}_2\} = \mathbb{Z}_n$.

(ii) If two half starters $v\,(G_0)$ and $v\,(G_1)$ are orthogonal, then $\Gamma = \{G_{x,i} : (x,i) \in \mathbb{Z}_n \times \mathbb{Z}_2\}$ with $G_{x,i} = (G_i + x)$ is an ODC of $K_{n,n}$.

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{u_0v_1 : v_0u_1 \in E(G)\}$ is called the *symmetric graph* of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a *symmetric starter* with respect $\mathbb{Z}_n$ if $v\,(G)$ and $v\,(G_s)$ are orthogonal.

(iii) Let n be a positive integer and let G a half starter represented by

$v\,(G) = (v_0,v_1,\ldots,v_{n-1}) \in \mathbb{Z}_n$. Then G is symmetric starter if and only if $\{v_i - v_{-i}+i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

El Shanawany et al. In [4] got some results for ODCs by using the cartesian product of two symmetric starter vectors, after proving the following Theorem.

Theorem 1 *The Cartesian product of any two symmetric starter vectors is a symmetric starter vector with respect to the Cartesian product of the corresponding groups.*

All our results in section 3 based on the following symmetric starter vectors of some graphs that can be used as ingredients for constructing an ODC of complete bipartite graphs by disjoint unions of graph-paths(that defined in section 2) by using the recursive constructing method called the cartesian product method.

1) P_{m+1} which is a symmetric starter of an ODC of $K_{m,m}$ whose vector is $v(P_{m+1})=(0,m-1,m-2^2,m-3^2,\ldots,m-3^2,m-2^2,m-1) \in \mathbb{Z}_m$, m is aprime number, see [5].
2) $nK_{2,2}$ which is a symmetric starter of an ODC of $K_{4n,4n}$ whose vector is $v(nK_{2,2})=(0,1,2,\ldots,2n-1,0,1,2,\ldots,2n-1) \in \mathbb{Z}_{4n}$, see Lemma 2.2.13 in [3].
3) $C_4 \cup K_{1,n-4}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is

$$v_i\,(C_4 \cup K_{1,n-4}) = \begin{cases} h+2 & : \ i = h-1,\ h, \text{ or} \\ h & : \ i = h+1,\ h+2, \text{ or} \\ h+1 & : \ \text{otherwise,} \end{cases}$$

where $n = 2h+1$, and n is odd, $n \geq 5$,

$$\text{and } v_i\,(C_4 \cup K_{1,n-4}) = \begin{cases} h & : \quad i = 1,\ 2h-1, \text{ or} \\ 0 & : \quad i = h-1,\ h+1, \text{ or} \\ 2h-i+1 & : \quad \text{otherwise,} \end{cases}$$

where $n = 2h$, and n is even, $n \geq 6$, see Lemma 4.1 in [6].

4) $C_8 \cup K_{1,n-8}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is

$$v_i\,(C_8 \cup K_{1,n-8}) = \begin{cases} 1 & : \quad i = 0, \text{ or} \\ 2 & : \quad i = 1,\ 3, \text{ or} \\ 8 & : \quad i = n-3,\ n-4, \text{ or} \\ 4 & : \quad i = n-1,\ n-2, \text{ or} \\ 1 & : \quad \text{otherwise,} \end{cases}$$

where $n \geq 9$, see Lemma 4.2 in [6].

5) $C_6 \cup K_{1,1} \cup K_{1,n-7}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is

$$v_i\,(C_6 \cup K_{1,1} \cup K_{1,n-7}) = \begin{cases} 1 & : \quad i = 0, \text{ or} \\ 4 & : \quad i = 1, n\text{-}2, \text{ or} \\ 0 & : \quad i = 2, 3, \text{ or} \\ 6 & : \quad i = n\text{-}3, n\text{-}1, \text{ or} \\ 3 & : \quad \text{otherwise,} \end{cases}$$

where $n \geq 7$, see Proposition 4.3 in [6].

6) $K_{1,2} \cup K_{1,2(n-1)}$ which is a symmetric starter of an ODC of $K_{2n,2n}$ whose vector is

$$v\,(K_{1,2} \cup K_{1,2(n-1)}) = (0, \overbrace{n, n, \ldots, n}^{(n\text{-}1)\text{ times}}, 0, \overbrace{n, n, \ldots, n}^{(n\text{-}1)\text{ times}}) \in \mathbb{Z}_{2n},\ n \geq 2, \text{ see [4].}$$

7) $\frac{n}{2} K_{1,2}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is

$v\,(\frac{n}{2} K_{1,2}) = (0,1,2,\ldots,n\text{-}1) \in \mathbb{Z}_n$, where $n \equiv 2,4 \bmod 6$, see Lemma 2.2.11 in [3].

8) $2K_{1,1} \cup K_{1,n-2}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is

$v\,(2K_{1,1} \cup K_{1,n-2}) = (0,0,\ldots,0,1,n\text{-}1) \in \mathbb{Z}_n$, where $n \geq 5$, see Lemma 2.2.17 in [3].

2. Graph-Path Definition

Let H be a certain graph, the graph H-Path denoted by $\mathbb{P}_{m+1}(H)$, is a path of set of vertices $\mathbb{V}=\{V_i : 0 \leq i \leq m\}$ and a set of edges $\mathbb{E} = \{ E_i : 0 \leq i \leq m\}$ if and only if there exists the following two bijective mappings:

1. $\phi : \mathbb{E} \rightarrow \mathcal{H}$ defined by $\phi\,(E_i) = H_i$, where $\mathcal{H} = \{ H_0, H_1, \ldots, H_{m-1}\}$ is a collection of m graphs each one is isomorphic to the graph H.
2. $\psi : \mathbb{V} \rightarrow \mathcal{A}$ defined by $\psi\,(V_i) = X_i$, where $\mathcal{A} = \{ X_i : 0 \leq i \leq m : \cap_i X_i = \varphi\}$ a class of disjoint sets of vertices.

In this paper, we are concerned with an ODC of $K_{n,n}$ by $\mathbb{P}_{m+1}(K_{\alpha,\beta})$, i.e. $H = K_{\alpha,\beta}$ is a subgraph of $K_{n,n}$, and for all $0 \leq i \leq m$,

$X_i = \{Y_{i/2} : i \equiv 0 \bmod 2\} \cup \{Z_{\lfloor i/2 \rfloor} : i \equiv 1 \bmod 2\}$, satisfying the following conditions

i. $|\,Y_{i/2}\,| = \alpha$ and $|\,Z_{\lfloor i/2 \rfloor}\,| = \beta\ \forall\ 0 \leq i \leq m$.

ii. $Y_{i/2} \subset \mathbb{Z}_n \times \{0\}$, $Z_{\lfloor i/2 \rfloor} \subset \mathbb{Z}_n \times \{1\}$, and $\cap_i Y_{i/2} = \cap_i Z_{\lfloor i/2 \rfloor} = \varphi$.

iii. for $0 \leq i \leq m\text{-}1$, $H_i \cong K_{\alpha,\beta}$ has the following edges:

$E(H_i)=\{ Y_{i/2} Z_{i/2} : i \equiv 0 \bmod 2\} \cup \{Y_{\lfloor i/2 \rfloor} Z_{\lceil i/2 \rceil}\} : i \equiv 1 \bmod 2\}$.

For more illustration, see Fig 1.

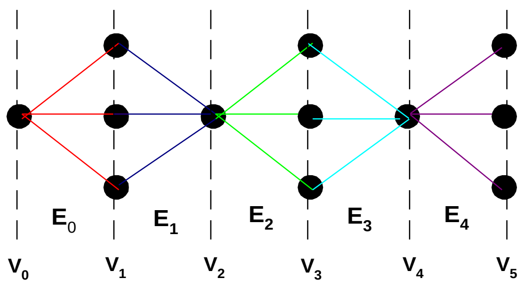

Fig 1. *$\mathbb{P}(K_{1,3})$, the path of 6 sets of vertices and 5 edges of $K_{1,3}$.*

3. ODCs of $K_{MN,MN}$ by Disjoint Unions of Graph-Paths

In the following, if there is no danger of ambiguity, if $(i_1,i_2) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, we can write (i_1,i_2) as (i_1i_2).

Let $v(G^j) = (v_0^j, v_1^j, \ldots, v_{n_j-1}^j) \in \mathbb{Z}_{n_j}^{n_j}$ be a symmetric starter vector of an ODC of K_{n_j,n_j}

by G^j with respect to $\mathbb{Z}_{n_j}$, where v_k^j, $k \in \mathbb{Z}_{n_j}$, $1 \le j \le 2$, then $v(G^1) \times v(G^2)$ is a symmetric starter vector of an ODC of $K_{n_1n_2,n_1n_2}$ by a new graph G with respect to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. The graph G can be described as follows : $E(G)=\{((v_{i_1}^1 v_{i_2}^2)_0,((v_{i_1}^1 v_{i_2}^2)+(i_1i_2))_1) : (i_1i_2) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, v_{i_1}^1 \in v(G^1),$

$v_{i_2}^2 \in v(G^2)\}$.

Theorem 2 *Let m be a prime number and* $\gcd(n,3)=1$, *then there exists an ODC of* $K_{mn,mn}$ *by* $n\mathbb{P}_{m+1}(K_{2,2})$.

Proof. Since $v(P_{m+1})$ and $v(nK_{2,2})$ are symmetric starter vectors, then $v(P_{m+1}) \times v(nK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_{4n}$ by applying Theorem 1. The resulting symmetric starter graph has the following edges set:

$E(n\mathbb{P}_{m+1}(K_{2,2}))=\{((v_{i_1}^1 v_{i_2}^2)_0,((v_{i_1}^1 v_{i_2}^2)+(i_1i_2))_1) : (i_1i_2) \in \mathbb{Z}_m \times \mathbb{Z}_{4n}, v_{i_1}^1 \in v(P_{m+1}), v_{i_2}^2 \in v(nK_{2,2})\}$. ■

For an illustration of Theorem 2, let $m = 3$ and $n = 2$, then there exists an ODC of $K_{24,24}$ by $2\mathbb{P}_4(K_{2,2})$ with respect to $\mathbb{Z}_3 \times \mathbb{Z}_8$, see Fig 2.

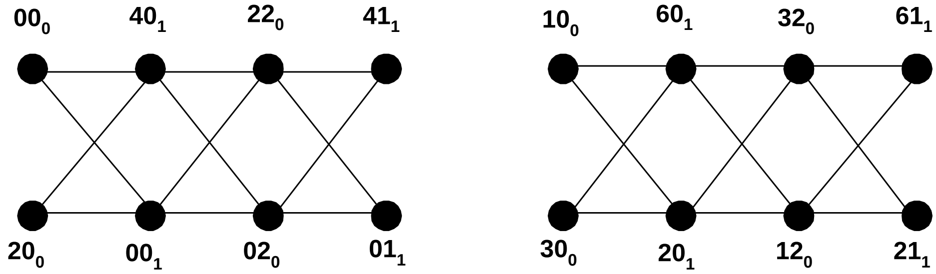

Fig 2. *Symmetric starter of an ODC of* $K_{24,24}$ *by* $2\mathbb{P}_4(K_{2,2})$ *with respect to* $\mathbb{Z}_3 \times \mathbb{Z}_8$.

Theorem 3 *Let m be a prime number and n* > 4, *then there exists an ODC of* $K_{mn,mn}$ *by* $\mathbb{P}_{m+1}(K_{2,2}) \cup \mathbb{P}_{m+1}(K_{1,n-4})$.

proof. Since $v(P_{m+1})$ and $v(C_4 \cup K_{1,n-4})$ are symmetric starter vectors, then $v(P_{m+1}) \times v(C_4 \cup K_{1,n-4})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_n$ by applying Theorem 1. The resulting symmetric starter graph has the following edges set: $E(\mathbb{P}_{m+1}(K_{2,2}) \cup \mathbb{P}_{m+1}(K_{1,n-4})) =$

$\{((v_{i_1}^1 v_{i_2}^2)_0,((v_{i_1}^1 v_{i_2}^2)+(i_1i_2))_1) : (i_1i_2) \in \mathbb{Z}_m \times \mathbb{Z}_n, v_{i_1}^1 \in v(P_{m+1}), v_{i_2}^2 \in v(C_4 \cup K_{1,n-4})\}$.

For an illustration of Theorem 3, let $m = 3$ and $n = 7$, then there exists an ODC of $K_{21,21}$ *by* $\mathbb{P}_4(K_{2,2}) \cup \mathbb{P}_4(K_{1,3})$ with respect to $\mathbb{Z}_3 \times \mathbb{Z}_7$, see Fig 3.

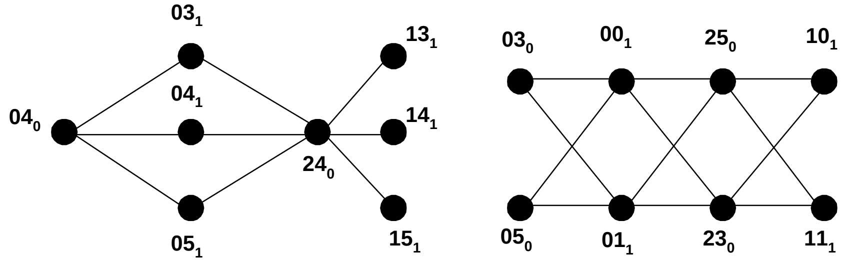

Fig 3. *Symmetric starter of an ODC of* $K_{21,21}$ *by* $\mathbb{P}_4(K_{2,2}) \cup \mathbb{P}_4(K_{1,3})$ *with respect to* $\mathbb{Z}_3 \times \mathbb{Z}_7$.

Theorem 4 *Let m be a prime number and* $n > 8$, *then there exists an ODC of* $K_{mn,mn}$ *by* $\mathbb{P}_{m+1}(C_8) \cup \mathbb{P}_{m+1}(K_{1,n-8})$.

proof. Since $v\,(P_{m+1})$ and $v\,(C_8 \cup K_{1,n-8})$ are symmetric starter vectors, then $v\,(P_{m+1}) \times v\,(C_8 \cup K_{1,n-8})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_n$ by applying Theorem 1. The resulting symmetric starter graph has the edges set: $E(\mathbb{P}_{m+1}(C_8) \cup \mathbb{P}_{m+1}(K_{1,n-8})) = \{((v^1_{i_1} v^2_{i_2})_0, ((v^1_{i_1} v^2_{i_2}) + (i_1 i_2))_1) : (i_1 i_2) \in \mathbb{Z}_m \times \mathbb{Z}_n, v^1_{i_1} \in v\,(P_{m+1}), v^2_{i_2} \in v\,(C_8 \cup K_{1,n-8})\}$. ■

For an illustration of Theorem 4, let $m = 2$ and $n = 9$, then there exists an ODC of $K_{18,18}$ by $\mathbb{P}_3(C_8) \cup \mathbb{P}_3(K_{1,1})$ with respect to $\mathbb{Z}_2 \times \mathbb{Z}_9$, see Fig 4.

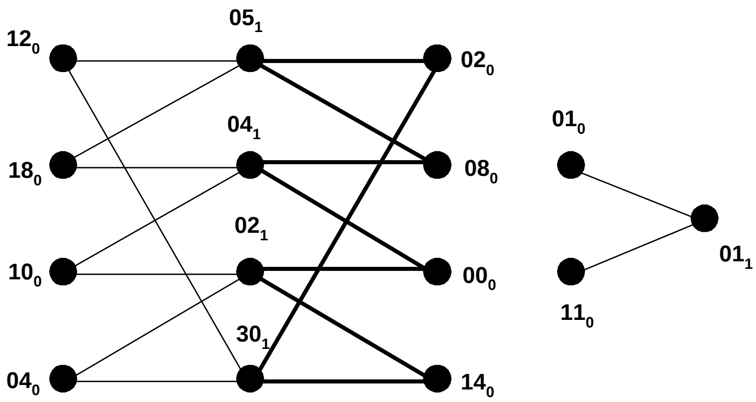

Fig 4. *Symmetric starter of an ODC of* $K_{18,18}$ *by* $\mathbb{P}_3(C_8) \cup \mathbb{P}_3(K_{1,1})$ *with respect to* $\mathbb{Z}_2 \times \mathbb{Z}_9$.

Theorem 5 *Let m be a prime number and* $n \geq 7$, *then there exists an ODC of* $K_{mn,mn}$ *by* $\mathbb{P}_{m+1}(C_6) \cup \mathbb{P}_{m+1}(K_{1,1}) \cup \mathbb{P}_{m+1}(K_{1,n-7})$.

Proof. Since $v\,(P_{m+1})$ and $v\,(C_6 \cup K_{1,1} \cup K_{1,n-7})$ are symmetric starter vectors, then $v\,(P_{m+1}) \times v\,(C_6 \cup K_{1,1} \cup K_{1,n-7})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_n$ by applying Theorem 1. The resulting symmetric starter graph has the following edges set:

$E(\mathbb{P}_{m+1}(C_6) \cup \mathbb{P}_{m+1}(K_{1,1}) \cup \mathbb{P}_{m+1}(K_{1,n-7}))) = \{ ((v^1_{i_1} v^2_{i_2})_0, ((v^1_{i_1} v^2_{i_2}) + (i_1 i_2))_1) : (i_1 i_2) \in \mathbb{Z}_m \times \mathbb{Z}_n,$

$v^1_{i_1} \in v\,(P_{m+1}), v^2_{i_2} \in v\,(C_6 \cup K_{1,1} \cup K_{1,n-7})\}$.

For an illustration of Theorem 5, let $m = 2$ and $n = 8$, then there exists an ODC of $K_{16,16}$ by $\mathbb{P}_3(C_6) \cup \mathbb{P}_3(K_{1,1}) \cup \mathbb{P}_3(K_{1,1})$ with respect to $\mathbb{Z}_2 \times \mathbb{Z}_8$, see Fig 5.

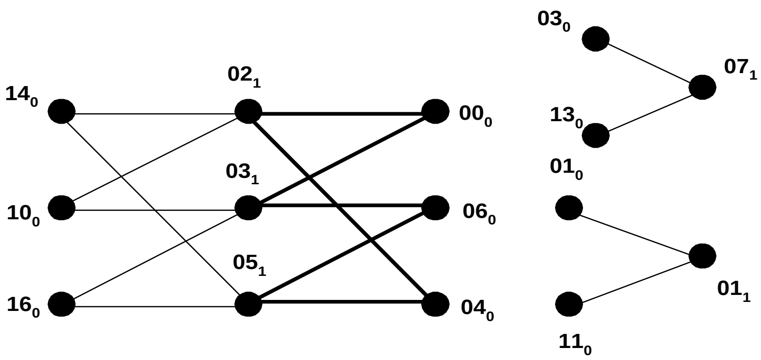

Fig 5. *Symmetric starter of an ODC of* $\mathbb{P}_3(C_6) \cup \mathbb{P}_3(K_{1,1}) \cup \mathbb{P}_3(K_{1,1})$ *with respect to* $\mathbb{Z}_2 \times \mathbb{Z}_8$.

Theorem 6 *Let m be a prime number and* $n \geq 2$, *then there exists an ODC of* $K_{2mn,2mn}$ *by* $\mathbb{P}_{m+1}(K_{1,2}) \cup \mathbb{P}_{m+1}(K_{1,2(n-1)})$.

Proof. Since $v\,(P_{m+1})$ and $v\,(K_{1,2} \cup K_{1,2(n-1)})$ are symmetric starter vectors, then $v\,(P_{m+1}) \times v\,(K_{1,2} \cup K_{1,2(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ by applying Theorem 1. The resulting symmetric starter graph has the following edges set:

$E(\mathbb{P}_{m+1}(K_{1,2}) \cup \mathbb{P}_{m+1}(K_{1,2(n-1)})) =\{ ((v_{i_1}^1 v_{i_2}^2)_0,((v_{i_1}^1 v_{i_2}^2)+(i_1 i_2))_1) : (i_1 i_2) \in \mathbb{Z}_m \times \mathbb{Z}_{2n},\ v_{i_1}^1 \in v\ (P_{m+1}),$ $v_{i_2}^2 \in v\ (K_{1,2} \cup K_{1,2(n-1)})\}$.

For an illustration of Theorem 6, let $m = 3$ and $n = 3$, then there exists an ODC of $K_{18,18}$ by $\mathbb{P}_4(K_{1,2}) \cup \mathbb{P}_4(K_{1,4})$ with respect to $\mathbb{Z}_3\times\mathbb{Z}_6$, see Fig 6.

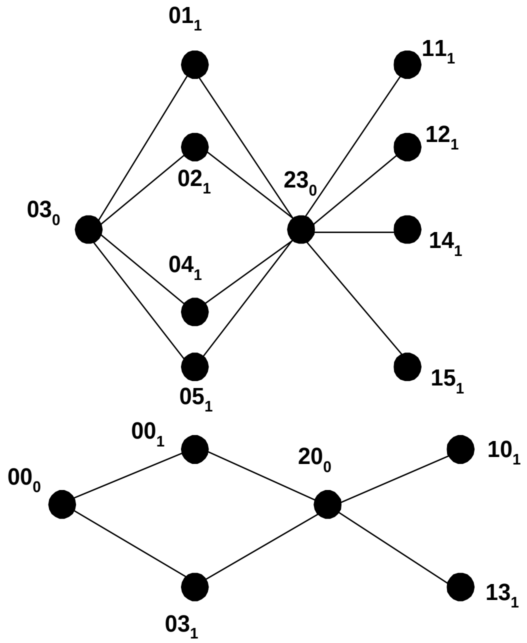

Fig 6. *Symmetric starter of an ODC of $K_{18,18}$ by $\mathbb{P}_4(K_{1,2}) \cup \mathbb{P}_4(K_{1,4})$ with respect to $\mathbb{Z}_3\times\mathbb{Z}_6$.*

Theorem 7 *Let m be a prime number and $n \equiv 2,4 \bmod 6$, then there exists an ODC of $K_{mn,mn}$ by* $(\frac{n}{2})\mathbb{P}_4(K_{1,2})$.

Proof. Since $v\ (P_{m+1})$ and $v\ (\frac{n}{2}K_{1,2})$ are symmetric starter vectors, then $v\ (P_{m+1}) \times v\ (\frac{n}{2}K_{1,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_n$ by applying Theorem 1. The resulting symmetric starter graph has the following edges set:

$E((\frac{n}{2})\mathbb{P}_4(K_{1,2})) =\{ ((v_{i_1}^1 v_{i_2}^2)_0,((v_{i_1}^1 v_{i_2}^2)+(i_1 i_2))_1) : (i_1 i_2) \in \mathbb{Z}_m \times \mathbb{Z}_n,\ v_{i_1}^1 \in v\ (P_{m+1}),$

$v_{i_2}^2 \in v\ (\frac{n}{2}K_{1,2})$.

For an illustration of Theorem 7, let $m = 3$ and $n = 8$, then there exists an ODC of $K_{24,24}$ by $4\mathbb{P}_4(K_{1,2})$ with respect to $\mathbb{Z}_3\times\mathbb{Z}_8$, see Fig 7.

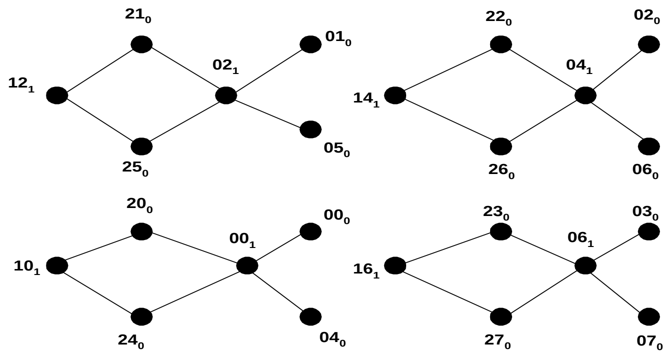

Fig 7. *Symmetric starter of an ODC of $K_{24,24}$ by $4\mathbb{P}_4(K_{1,2})$ with respect to $\mathbb{Z}_3 \times \mathbb{Z}_8$.*

Theorem 8 *Let m be a prime number and n* ≥ 5, *then there exists an ODC of* $K_{mn,mn}$ *by* $2\mathbb{P}_{m+1}(K_{1,1}) \cup \mathbb{P}_{m+1}(K_{1,n-2})$.

Proof. Since $v\ (P_{m+1})$ and $v\ (2K_{1,1} \cup K_{1,n-2})$ are symmetric starter vectors, then $v\ (P_{m+1}) \times v\ (2K_{1,1} \cup K_{1,n-2})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_n$ by applying Theorem 1. The resulting symmetric starter graph has the following edges set:

$E(2\mathbb{P}_{m+1}(K_{1,1}) \cup \mathbb{P}_{m+1}(K_{1,n-2})) = \{ ((v_{i_1}^1 v_{i_2}^2)_0, ((v_{i_1}^1 v_{i_2}^2)+(i_1 i_2))_1) : (i_1 i_2) \in \mathbb{Z}_m \times \mathbb{Z}_n,\ v_{i_1}^1 \in v\ (P_{m+1}),$

$v_{i_2}^2 \in v\ (2K_{1,1} \cup K_{1,n-2}) \}$.

For an illustration of Theorem 8, let $m = 3$ and $n = 5$, then there exists an ODC of $K_{15,15}$ by $2\mathbb{P}_4(K_{1,1}) \cup \mathbb{P}_4(K_{1,3})$ with respect to $\mathbb{Z}_3 \times \mathbb{Z}_5$, see Fig 8.

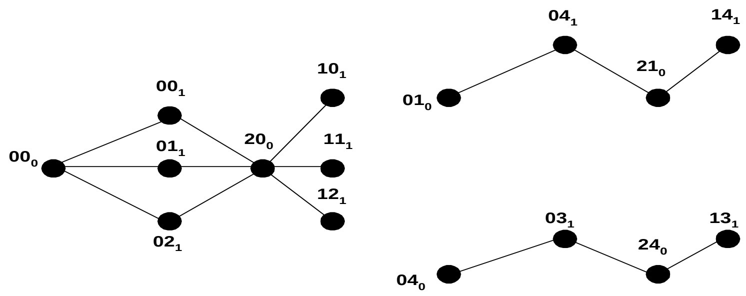

Fig 8. *Symmetric starter of an ODC of* $K_{15,15}$ *by* $2\mathbb{P}_4(K_{1,1}) \cup \mathbb{P}_4(K_{1,3})$ *with respect to* $\mathbb{Z}_3 \times \mathbb{Z}_5$.

4. Conclusion

Using the Cartesian product of the two symmetric starter vectors, $v\ (P_{m+1}) \in \mathbb{Z}_m$ and $v\ (G) \in \mathbb{Z}_n$, we can get $v\ (\mathbb{P}_{m+1}(G))$ which is considered a new symmetric starter vector of an ODC of $K_{mn,mn}$.

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