*International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*
*Volume 2, Issue 3, March 2014, PP 256-261*
*ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)*
*www.arcjournals.org*

# Properties of $P_0$ – Almost Distributive Lattices

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**Abstract:** *Characterizations of an Almost Distributive Lattice with a maximal and least n-term chain base are derived.*

**Keywords:** *Almost Distributive Lattice(ADL); Birkhoff Center; Maximal element; Chain base; $P_0$ – lattice; $P_0$ – Almost Distributive Lattice( $P_0$ – ADL).*

## 1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by U.M. Swamy and G.C. Rao[8] as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. The concept of Birkhoff center ***B*** of an ADL ***A*** was introduced in [9] and it was observed that ***B*** is a relatively complemented ADL.

G. Epstein and A. Horn introduced the concept of a $P_0$ – lattice in [5]. Later, T. Traczyk, Ph. Dwinger are studied and explored its properties. $P_0$ – lattice has good applications in computers and logic on the lines of G. Epstein and A. Horn [3,4]. For this reason, G.C. Rao extended this concept in to the class of ADLs as $P_0$ – Almost Distributive Lattices as a generalization of $P_0$ – lattice. In this paper, we derive some important properties of $P_0$ – Almost Distributive Lattice. These properties will help the further investigations of possible applications of $P_1$ – Almost Distributive Lattices, $P_2$ – Almost Distributive Lattices and Post Almost Distributive Lattice in logic and computer science on the lines of G. Epstein and A. Horn [3,4].

## 2. Preliminaries

In this section, we give the necessary definitions and important properties of an ADL taken from [8] for ready reference.

**Definition 1.1** [8] An algebra $(A,\vee,\wedge,0)$ of type $(2,2,0)$ is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:

i. $x \vee 0 = x$

ii. $0 \wedge x = 0$

iii. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

iv. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

v. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

vi. $(x \vee y) \wedge y = y$, for all $x, y, z \in A$.

**Theorem 1.2.** [8] Let $m$ be a maximal element in an ADL $A$ and $x \in A$. Then the following are equivalent:

i. $x$ is a maximal element of $(A, \leq)$.

ii. $x \wedge m = m$.

iii. $x \wedge a = a$, for all $a \in A$.

iv. $x \vee a = x$, for all $a \in A$.

v. $a \vee x$ is maximal, for all $a \in A$.

**Definition 1.3. [9]** Let $A$ be an ADL with a maximal element $m$ and

$B(A) = \{x \in A \mid x \wedge y = 0$ and $x \vee y$ is maximal for some $y \in A\}$. Then $(B(A), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of $A$. We use the symbol $B$ instead of $B(A)$ when there is no ambiguity.

For other properties of Birkhoff center of an ADL, we refer [9].

In our paper [7], we introduced the concept of Pseudo-supplemented Almost Distributive Lattices and derive its properties. The following definition was taken from [7].

**Definition 1.4. [7]** Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. $A$ is called a Pseudo-supplemented Almost Distributive Lattice (or, simply a PSADL) if, for each $x \in A$, there exists $b \in B$ such that

$P_1$: $x \wedge b = b$

$P_2$: if $c \in B$ and $x \wedge c = c$, then $b \wedge c = c$.

Here $b \wedge m$ is uniquely determined by $x$ and it is denoted by $x!$, the pseudo-supplement of $x$. Also, we observe that $x! \in B([0, m])$. For other properties of PSADL, we refer [7].

## 3. PROPERTIES OF $P_0$ – ADLS

The concept of $P_0$ – lattice was introduced by G. Epstein and A. Horn in [5]. The following definition is taken from [5].

**Definition 2.1.** [5] Let $A$ be a bounded distributive lattice and let $B$ be a Boolean subalgebra of the center of $A$. A chain base of $A$ is a finite sequence $0 = e_0 \leq e_1 \leq e_2 \leq \ldots\ldots \leq e_{n-1} = 1$ such that $A$ is generated $B \cup \{e_o, e_1, e_2, \ldots\ldots, e_{n-1}\}$. If $A$ has a chain base, then $A$ is called a $P_0$ – lattice.

The concept of $P_0$ – Almost Distributive Lattice ($P_0$ – ADL) was introduced by G.C. Rao and A. Meherat in [6] as follows.

**Definition 2.2.** [6] Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. Then $A$ is a $P_0$ – Almost Distributive Lattice(or, simply a $P_0$ – ADL) if and only if there exist elements $0 = e_0, e_1, e_2, \ldots., e_{n-1}$ in $A$ such that:

i. $e_{n-1} \wedge m = m$

ii. $e_i \wedge e_{i-1} = e_{i-1}$, for $1 \leq i \leq n-1$

iii. for any $x \in A$, there exist $b_i \in B$ such that $x \wedge m = \bigvee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)$.

A set $\{0 = e_0, e_1, e_2, \ldots\ldots, e_{n-1}\}$ of elements in a $P_0$ – ADL $A$ satisfying conditions $(i),(ii)$ and $(iii)$ is called a chain base of $A$.

**Definition 2.3** [6] Let $(A; e_0, e_1, e_2, \ldots.., e_{n-1})$ is a $P_0$ – ADL and $x \in A$ such that $x \wedge m = \bigvee_{i=1}^{n-1} (b_i \wedge e_i \wedge m)$. ............(♦) where $b_i \in B$.

i. If $b_i \wedge b_{i+1} = b_{i+1}$ for $1 \le i \le n-2$, then (♦) is called a monotone representation of $x$, or simply as mono. rep.

ii. If $b_i \wedge b_j = 0$ for $i \neq j$, then (♦) is called a disjoint representation of $x$, or simply as dis. rep.

We observed that every element in $A$ has both a mono. and dis. representation. The following theorem is easily proved by induction.

**Theorem 2.4.** Let $(A, \vee, \wedge, 0, m)$ be and ADL with a maximal element $m$ and Birkhoff center Let $b_i, e_i \in A$ for $0 \le i \le n-1$ such that $b_i = b_{i-1} \wedge b_i$ and $e_{i-1} = e_i \wedge e_{i-1}$. Then,

$$\bigvee_{i=0}^{n-1} (b_i \wedge e_i \wedge m) = b_0 \wedge e_{n-1} \wedge \bigwedge_{i=1}^{n-1} (b_i \vee e_{i-1}) \wedge m.$$

Here afterwards, $(A; e_0, e_1, e_2, \ldots.., e_{n-1})$ stands for a $P_0$ – ADL $(A, \vee, \wedge, 0, m)$ with a chain base $\{0 = e_0, e_1, e_2, \ldots.., e_{n-1}\}$ and Birkhoff center $B$.

Now we prove the following.

**Theorem 2.5.** Let $(A; e_0, e_1, e_2, \ldots.., e_{n-1})$ be a $P_0$ – ADL. Then $A$ has a maximal $n$ – term chain base $\{0 = e_0 \le e_1 \le e_2 \le \ldots.. \le e_{n-1}\}$ if and only if $b \wedge e_i \wedge m \le e_{i-1} \wedge m$ implies $b \wedge m \le e_{i-1} \wedge m$ for $b \in B$ and $1 \le i \le n-1$.

**Proof:** Let $e$ be the maximal chain base in $P_0$ – ADL $A$. Suppose $b \wedge e_i \wedge m \le e_{i-1} \wedge m$ for $b \in B$ and $1 \le i \le n-1$.

Let $f_i \wedge m = (e_i \vee (b \wedge e_{i+1})) \wedge m$. Then

$b^m \wedge f_i \wedge m = b^m \wedge (e_i \vee (b \wedge e_{i+1})) \wedge m$ where $b^m$ is the complement of $b \wedge m$ in [0,m]

$$= (b^m \wedge e_i \wedge m) \vee (b^m \wedge e_{i+1} \wedge m)$$

$$= b^m \wedge e_i \wedge m.$$

Since $b \in B$, we have $e_i \wedge m = (b \vee b^m) \wedge e_i \wedge m$

$$= (b \wedge e_i \wedge m) \vee (b^m \wedge e_i \wedge m)$$

$$\le (e_{i-1} \wedge m) \vee (b^m \wedge f_i \wedge m)$$

$$\le (e_i \vee f_i) \wedge m$$

$$\le f_i \wedge m.$$

Since $e_i \wedge m \le f_i \wedge m \le e_{i+1} \wedge m$, weget

$0 = e_0, e_1 \wedge m, e_2 \wedge m, \ldots., e_{i-1} \wedge m \le f_i \wedge m \le e_{i+1} \wedge m, e_{i+2} \wedge m, \ldots\ldots., e_{n-1} \wedge m$ is a chain base of $A$. Since $f_i \wedge m \ge e_i \wedge m$ and $e$ is the maximal chain base of $A$, we conclude that

$f_i \wedge m = e_i \wedge m$. Then $b \wedge e_i \wedge m \leq e_i \wedge m$ and hence $b \wedge e_{i+1} \wedge m \leq b \wedge e_i \wedge m \leq e_{i-1} \wedge m$. Repeating the above argumentation a finite number of times, we get $b \wedge e_{n-1} \wedge m \leq e_{i-1} \wedge m$ and hence $b \wedge m \leq e_{i-1} \wedge m$. Conversely, suppose that $b \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $b \wedge m \leq e_{i-1} \wedge m$ for $b \in B$ and $1 \leq i \leq n-1$. Suppose $e$ and $f$ are two $n-$ term chain bases of $A$ and $b \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $b \wedge m \leq e_{i-1} \wedge m$ for $b \in B$ and $1 \leq i \leq n-1$. By our assumption and Theorem 3.18[6], we get that $A$ is a pseudo-complemented ADL and again, by Theorem 3.21[6], we get $A$ has a chain base $\{g_0, g_1, g_2, ......, g_{n-1}\}$ such that $g_1$ is the smallest dense element of $A$. It is enough to show that $f \leq e$ in the case $f_1$ is the smallest dense element of $A$. Consider

$$e_1 = (a_{i1} f_1 \vee a_{i2} f_2 \vee a_{i3} f_3 \vee ....... \vee a_{i,n-1} f_{n-1}) \wedge m$$

$$f_1 = (b_{i1} f_1 \vee b_{i2} f_2 \vee b_{i3} f_3 \vee ....... \vee b_{i,n-1} f_{n-1}) \wedge m \quad \text{for } i = 1, 2, ....., n-1.$$

Then $a_{11}^m \wedge e_1 = 0$ where $a_{11}^m$ is the complement of $a_{11} \wedge m$ in $[0, m]$ and hence $a_{11} \wedge m = m$. Similarly, we get $b_{11} \wedge m = m$, being $f_1$ is the smallest dense element of $A$. Thus $e_1 \leq f_1$. Similarly, we get $f_1 \leq e_1$ and hence $e_1 = f_1$. Clearly $\{e_0, e_1, ....., e_{n-1}\}$ and $\{f_0, f_1, ....., f_{n-1}\}$ are chain bases in $[e_1, e_{n-1}]$ and members of the center of this interval are of the form $(e_1 \vee (b \wedge e_{n-1})) \wedge m$. Then, by the hypothesis, we get $(b \vee e_1) \wedge m \leq e_{i-1} \wedge m$ for $i \geq 2$. By applying the above argumentation may be applied to this case as well finite number of steps, we get that $A$ has a maximal $n-$ term chain base. ●

**Theorem 2.6.** Let $(A; e_0, e_1, e_2, ....., e_{n-1})$ be a $P_0-$ ADL. Then the following are equivalent:

i. $b \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $b \wedge e_i = 0$ for every $b \in B$ and $1 \leq i \leq n-1$.

ii. $(b \vee e_{i-1}) \wedge m \geq e_i \wedge m$ implies $b \wedge m \geq e_i \wedge m$ for every $b \in B$ and $1 \leq i \leq n-1$.

**Proof.** Let $b \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $b \wedge e_i = 0$ for every $b \in B$ and $1 \leq i \leq n-1$.

Suppose $(b \vee e_{i-1}) \wedge m \geq e_i \wedge m$.

Then $b^m \wedge (b \vee e_{i-1}) \wedge m \geq b^m \wedge e_i \wedge m$ where $b^m$ is complement of $b \wedge m$ in $[0, m]$

$\Rightarrow b^m \wedge e_{i-1} \wedge m \geq b^m \wedge e_i \wedge m$

$\Rightarrow b^m \wedge e_i \wedge m \leq e_{i-1} \wedge m \leq e_i \wedge m$

$\Rightarrow (b \vee (b^m \wedge e_i)) \wedge m \leq b \wedge e_i \wedge m$

$\Rightarrow (b \vee e_i) \wedge m \leq b \wedge m$

$\Rightarrow e_i \wedge m \leq b \wedge m$.

Thus we get (i)$\Rightarrow$ (ii). Similarly, we get (ii)$\Rightarrow$ (i).

In the following theorem, we derive some important properties of a $P_0-$ ADL and using this theorem, we can replace $x \in B$ in the hypothesis of Theorem 3.5 by $x \in A$.

**Theorem 2.7.** Let $(A; e_0, e_1, ...., e_{n-1})$ be a $P_0-$ ADL with a maximal element $m$ and Birkhoff center $B$. Then the following are equivalent:

i. $b \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $b \wedge m \leq e_{i-1} \wedge m$ for every $b \in B, 1 \leq i \leq n-1$.

ii. $x \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $x \wedge m \leq e_{i-1} \wedge m$ for every $x \in A, 1 \leq i \leq n-1$.

**Proof.** Clearly (i) follows (ii). Now assume $x \wedge e_i \wedge m \leq e_{i-1} \wedge m$ implies $x \wedge m \leq e_{i-1} \wedge m$ for every $x \in A, 1 \leq i \leq n-1$. Suppose $x \in A$ and $x \wedge e_{i-1} \wedge m \leq e_i \wedge m$. Since $x \in A, x$ has a mono. representation. Let $x \wedge m = ê_{i=1}^{n-1}(x_i \wedge e_i \wedge m)$. Then $x_i \wedge e_i \wedge m \leq e_{i-1} \wedge m$ and hence $x \wedge e_i \wedge m \leq e_{i-1} \wedge m$ and hence $x_i \wedge m \leq e_{i-1} \wedge m$. By monotonicity gives $x_j \wedge m \leq e_{i-1} \wedge m$ for $j \leq i$. Therefore $x_j \wedge e_j \wedge m \leq e_{i-1} \wedge m$ for $j \leq i$. Similarly, we get $x_j \wedge e_j \wedge m \leq e_{i-1} \wedge m$ for $j \leq i$. Hence $x \wedge m \leq e_{i-1} \wedge m$.

Unlike in lattices, the dual of an ADL in not an ADL, in general. For this reason, we introduce the concept of a dual $P_0$ – Almost Distributive Lattice as a generalization of a dual $P_0$ – lattice.

**Definition. 2.8.** Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. Then $A$ is said to be a dual $P_0$ – Almost Distributive Lattice (or, simply a dual $P_0$ – ADL) if and only if there exist elements $0 = e_0, e_1, e_2, ...., e_{n-1} \in A$ such that:

i. $e_{i-1} \wedge e_i = e_i$, for $1 \leq i \leq n-1$

ii. for any $x \in A$, there exist $b_i \in B$ such that $x \wedge m = ì_{i=1}^{n-1}(b_i \vee e_i) \wedge m$.

We observed that if $(A; e_0, e_1, ...., e_{n-1})$ is a dual $P_0$ – ADL with a maximal element $m$ and Birkhoff center $B$, then $b \wedge e_{i-1} \wedge m \leq e_i \wedge m$ implies $b \wedge m \leq e_i \wedge m$ for every $b \in B$ and $1 \leq i \leq n-1$. Finally, we conclude this paper with the following theorem and it is derive directly from Theorem 2.5 and Theorem 2.7.

**Theorem 2.9.** If $(A; e_0, e_1, ...., e_{n-1})$ be a dual $P_0$ – ADL with a maximal element $m$ and Birkhoff center $B$ is the least chain base of a $P_0$ – ADL, then $e_i!$ exists and equals to $0$ for $i = 1, 2, ... n-2$.

## REFERENCES


[1] Birkhoff, G.: Lattice Theory.,Amer. Math. Soc. Colloq. Publ. XXV, Providence(1967), U.S.A.

[2] G. Epstein and A. Horn : Chain based lattices, Pacific Journal of Mathematics,Vol. 55, No. 1, 1974.

[3] G. Epstein and A. Horn : A propositional calculus for affirmation and negation with linearly ordered matrix(abstract), J. Symb. Logic, 1972, Vol 37, 439.

[4] G. Epstein and A. Horn : Propositional calculi based on sub-residuation(abstract),J. Symb. Logic, 1973, Vol 38, 546-547.

[5] Epstein, G. and Horn, A.: P-algebras, an abstraction from Post algebras, Vol 4,Number 1,195-206,1974, Algebra Universalis.

[6] Rao, G.C. and Meherat Alamneh: P0-Almost Distributive Lattices, Accepted for publication in Southeast Asian Bulletin of Mathematics.

[7] Rao, G.C. and Naveen Kumar Kakumanu, Pseudo-Supplemented Almost Distributive Lattices, Southeast Asian Bulletin of Mathematics, Vol 37(2013),131 to 138.

[8] Swamy, U.M. and Rao, G.C., Almost Distributive Lattices, J. Aust. Math. Soc. (SeriesA), Vol.31 (1981), 77-91.

[9] Swamy, U.M. and Ramesh, S., Birkhoff center of ADL, Int. J. Algebra, Vol.3 (2009), 539-546.

[10] T. Traczyk, Axioms and Properties of Post algebra, Coll. Math., Vol X, 1963, pp 193-200.



## ACKNOWLEDGEMENTS

This paper is dedicated to my beloved wife Smt. K. Ramanawin.

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