International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 3, March 2014, PP 322-329
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

The regular Property of Duo Semigroups and Duo Rings

Mohammadi Begum

Research Scholar,
Sarva Bharatiya Shiksha Peeth, India.

Abstract: *In mathematics, a semigroup is an algebraic structure that consists of a set together with an associative binary operation. A semigroup generalizes a monoid. As such, there might not exist an identity element. It also generalizes a group (a monoid with all inverses) to a type where every element need not have an inverse, which ascribes the name semigroup. The concept of a semigroup is very simple, but plays a predominant role in the development of Mathematics. This paper discusses in detail the regular property of duo semigroups and duo rings.*

1. INTRODUCTION

The study of semigroups trailed behind that of other algebraic structures with more complex axioms such as groups or rings. A number of sources attribute the first use of the term (in French) to J.-A. de Séguier in Élements de la Théorie des Groupes Abstraits (Elements of the Theory of Abstract Groups) in 1904. The term is used in English in 1908 in Harold Hinton's Theory of Groups of Finite Order.

Anton Suschkewitsch obtained the first non-trivial results about semigroups. His 1928 paper Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit (On finite groups without the rule of unique invertibility) determined the structure of finite simple semigroups and showed that the minimal ideal (or Green's relations J-class) of a finite semigroup is simple. From that point on, the foundations of semigroup theory were further laid by David Rees, James Alexander Green, Evgenii Sergeevich Lyapin, Alfred H. Clifford and Gordon Preston. The latter two published a two-volume monograph on semigroup theory in 1961 and 1967 respectively. In 1970, a new periodical called Semigroup Forum (currently edited by Springer Verlag) became one of the few mathematical journals devoted entirely to semigroup theory.

The algebraic theory of semigroups was developed by Clifford and Preston; Petrich and Ljapin. The ideal theory in commutative semigroups has developed by Bourne, Harbans Lal, Satyanarayana, Mannepalli and Nagore.

In recent years researchers in the field have become more specialized with dedicated monographs appearing on important classes of semigroups, like inverse semigroups, as well as monographs focusing on applications in algebraic automata theory, particularly for finite automata, and also in functional analysis, duo semigroups, duo chained semigroups and noetherian groups.

Regular semigroups were introduced by J. A. Green in his influential 1951 paper "On the structure of semigroups"; this was also the paper in which Green's relations were introduced. The concept of regularity in a semigroup was adapted from an analogous condition for rings, already considered by J. von Neumann. It was his study of regular semigroups which led Green to define his celebrated relations. According to a footnote in Green 1951, the suggestion that the notion of regularity be applied to semigroups was first made by David Rees. Regular semigroups are one of the most-studied classes of semigroups, and their structure is particularly amenable to study via Green's relations. The motivation for this paper is to understand the structure of a duo semigroup. Structure of semigroups has been well studied, however, the structure of duo semigroups has not been studied as deeply. This paper specifically investigates the regular property of duo semigroups and duo rings and analyzes how the regularity influences the structure of duo semigroups.

2. DUO SEMIGROUPS AND DUO RINGS

Definitions 2.1

A semigroup S is said to be a left duo semigroup if every left ideal of S is a two sided ideal of S. A semigroup S is said to be a right duo semigroup if every right ideal of S is a two sided ideal of S. A semigroup S is said to be a duo semigroup if it is both a left duo semigroup and a right duo semigroup.

Definition 2.2

A regular semigroup is a semigroup S in which every element is regular, i.e., for each element a, there exists an element x such that $axa = a$.

There are two equivalent ways in which to define a regular semigroup S:

(1) for each a in S, there is an x in S, which is called a **pseudoinverse**, with $axa = a$;

(2) every element a has at least one **inverse** b, in the sense that $aba = a$ and $bab = b$.

Let us see the equivalence of the above two Definitions.

First suppose that S is defined by (2).

Then b serves as the required x in (1).

Conversely, if S is defined by (1), then xax is an inverse for a,

since $a(xax)a = axa(xa) = axa = a$

and $(xax)a(xax) = x(axa)(xax) = x(axa)x = xax$.

A regular semigroup in which idempotents commute is an inverse semigroup, that is, every element has a unique inverse.

To see this, let S be a regular semigroup in which idempotents commute.

Then every element of S has at least one inverse.

Suppose that a in S has two inverses b and c, i.e.,

$aba = a$, $bab = b$, $aca = a$ and $cac = c$. Also ab, ba, ac and ca are idempotents as above.

Then

$b = bab = b(aca)b = bac(a)b = bac(aca)b = bac(ac)(ab) = bac(ab)(ac) = ba(ca)bac = ca(ba)bac = c(aba)bac = cabac = cac = c$.

So, by commuting the pairs of idempotents ab & ac and ba & ca, the inverse of a is shown to be unique.

Conversely, it can be shown that any inverse semigroup is a regular semigroup in which idempotents commute.

Definition 2.3

An element a of a semigroup S is said to be ***regular*** if $a = a\,x\,a$ for some $x \in$ S.

Definition 2.4

An element a of a semigroup S is said to be ***completely regular*** if $a = a\,x\,a$ and $ax = xa$ for some $x \in$ S.

Definition 2.5

An element a of a semigroup S is said to be ***left regular*** if $a = a^2x$ for some $x \in$ S.

Definition 2.6

An element a of a semigroup S is said to be ***right regular*** if $a = xa^2$ for some $x \in$ S.

Definition 2.7

An element a of a semigroup S is said to be ***intra regular*** if $a = xa^2y$ for some $x, y \in$ S.

Theorem 2.8

Let S be a semigroup and $a \in$ S. If a is completely regular then a is regular.

Proof **:** Suppose a is completely regular of a semigroup S, then $a = a\,x\,a$, $ax = xa$ for some $x \in$ S, hence a is regular.

Theorem 2.9

Let S be a semigroup and $a \in$ S. Then a is completely regular if and only if a is both a left regular and a right regular.

Proof: Suppose a is completely regular. Then $a = axa$, $ax = xa$ for some $x \in$ S.

Now $a = axa = aax = a^2x$. Therefore a is left regular.

Also $a = axa = xaa = xa^2$. Therefore a is right regular.

Conversely suppose that a is both left regular and right regular.

Then $a = a^2x$, $a = ya^2$ for some $x,y \in$ S. Now $a = a^2x = a(a)x = a(ya^2)x = ay(a^2x) = aya$.

And $a = ya^2 = y(a)a = y(a^2x)a = (ya^2)xa = axa$.

Now $a(yax)a = (aya)xa = axa = a$ and $(yax)a(yax) = y(axa)yax = y(a)yax = y(aya)x = yax$. Also $a(yax) = (aya)x = ax = ya^2x = ya = y(axa) = (yax)a$.

Therefore a is completely regular

Definition 2.10

Let S be any semigroup. Then a non-empty set A is said to ***generate S*** (or) S is said to be ***generated by A*** if every element of S is a finite product of elements of A.

Definition 2.11

A non-empty subset A of a semigroup S is said to be a ***sub semigroup*** of S

if $a, b \in$ A implies $ab \in$ A.

Definition 2.12

A non-empty subset A of a semigroup S is said to be a ***left ideal*** if SA $\subseteq$ A.

Definition 2.13

A non-empty subset A of a semigroup S is said to be a ***right ideal*** if AS $\subseteq$ A.

Definition 2.14

A non-empty subset A of a semigroup S is said to be a ***two sided ideal*** or simply an ***ideal*** if it is both a left and a right ideal of S.

Definition 2.15

Let A be a nonempty subset of a semigroup S. Then the intersection of all (left, right) ideals of S containing A is called ***the (left,right) ideal generated by A***.

Definition 2.16

An ideal A of a semigroup S is said to be a ***principal ideal*** provided A is an ideal generated by single element set. If an ideal A is generated by a, then A is denoted as $< a >$ or J[a].

Definition 2.17

An element a of a semigroup S is said to be ***semisimple***

if $a \in\ < a >^2$, i.e., $< a >^2 = < a >$.

Definition2.18 An ideal A of a semigroup S is called a ***finnitely generated ideal*** if it is a union of finite number of principal ideals.

Theorem 2.19

Let S be a semigroup and $a \in$ S. **If** a **is left regular, then** a **is semisimple**

***Proof*:** Suppose a is left regular, then $a = a^2 x$ for some $x \in$ S and $a \in < a >^2$. Therefore a is semi simple.

Theorem 2.20

Let S be a semigroup and $a \in$ S. **If** a **is right regular, then** a **is semisimple.**

***Proof*:** Suppose a is right regular, then $a = xa^2$ for some $x \in$ S, $a \in < a >^2$.

Therefore a is semisimple.

Theorem 2.21

Let S be a semigroup and $a \in$ S. **If** a **is intra regular, then** a **is semisimple**

***Proof*:** a is intra regular $a = xa^2y$ for some $x, y \in$ S, $a \in < a >^2$.

Hence a is semisimple.

Theorem 2.22

Let S be a semigroup and $a \in$ S. **If** a **is completely regular then** a **is semi simple.**

***Proof*:** Suppose a is completely regular of a semigroup S, then $a = a\,x\,a$, $ax=xa$ for some $x \in$ S *and* $a \in < a >^2$, therefore a is semi simple.

Theorem 2.23

If a semigroup S is regular then every principal ideal of S is generated by an idempotent.

***Proof*:** Suppose S is a regular semigroup. Let $< a >$ be a principal ideal of S.

Since S is regular, there exists $x \in$ S such that $axa = a$. Now $(ax)^2 = axax = ax$.

Let ax = e then obviously $<a> = < e >$.

Therefore every principal ideal is generated by an idempotent.

Theorem 2.24

A semigroup S is a regular semigroup if and only it every principal ideal is generated by an idempotent.

***Proof*:** Suppose S is a regular semigroup, then $a = a\,x\,a$ for some $x \in$ S for all $a \in$ S.

Since $a \in$ S , $< a >$ be a principal ideal of S. Suppose $ax = b$ for some $b \in$ S.

Now $a = a\,x\,a = ba \in < b >$ implies $b = ax = ax\,a\,x \in < a >$.

Therefore $< a > = < b >$, implies every principal ideal is generated by an idempotent.

Conversely, suppose that every principal ideal of S generated by an idempotent, for each $a \in$ S , $< a > = < e >$ for some idempotent element e of S.

Definition 2.25

A semigroup S is said to be a ***completely regular semigroup*** if every element is completely regular

3. Results on Duo Semigroups

Theorem 3.1

In a duo semigroup S, an element x **is semisimple if and only if** x **is intraregular.**

Proof: Suppose that x is a semisimple element of S. Then $x \in \langle x \rangle^2$. Since S is a duo semigroup, by the fact that "Every pseudo symmetric semigroup is a semipseudo symmetric semigroup", every ideal of S is a semipseudo symmetric ideal and hence $\langle x^2 \rangle$ is semipseudo symmetric ideal. Thus $x^2 \in \langle x^2 \rangle$, $\langle x \rangle^2 \in \langle x^2 \rangle$,

$x \in \langle x \rangle^2 \langle x^2 \rangle$. Therefore $x = sx^2t$ for some $s, t \in$ S. Hence S is intraregular.

Conversely suppose that $x \in$ S is intraregular. Then $x = sx^2t$ for some $s, t \in S^1$ and hence $x \in \langle x \rangle^2$. Therefore x is semisimple.

Theorem 3.2

If S is a duo semigroup and $a \in$ S, then the following are equivalent.

1) *a* is completely regular.

2) *a* is regular .

3) *a* is left regular .

4) *a* is right regular.

5) *a* is intra regular .

6) *a* is semisimple.

Proof: Let S is a duo semigroup and $a \in$ S.

Then $aS^1a = a^2S^1 = S^1a^2 = \langle a \rangle^2 = \langle a \rangle^2$.

(1) $\Rightarrow$ (2): Suppose that a is completely regular.

By theorem 2.8, a is regular

(2) $\Rightarrow$ (3): Suppose that a is regular. Then $a = a x a$ for some $x \in$ S.

Now $a x a$ $aS^1a = a^2S^1$. Therefore $a = a^2y$ for some $y \in S^1$ and hence S is left regular.

(3)$\Rightarrow$ (4): Suppose that a is left regular. Then $a = a^2x$ for some $x \Rightarrow$ S. Now a^2x $a^2S^1 = S^1a^2$. Therefore $a = ya^2$ for some $y \in S^1$. Hence a is right regular.

(4) $\Rightarrow$ (5): Suppose that a is right regular. Then $a = xa^2$ for some $x \in$ S. Now $a \in S^1a^2 = \langle a \rangle^2$. Therefore $a \in \langle a^2 \rangle$. Thus $a = xa^2y$ for some $x, y \in S^1$. Hence a is intra regular.

(5) $\Rightarrow$ (6): Suppose that a is intra regular. Then $a = xa^2y$ for some $x,y \in$ S, and hence $a \in \langle a^2 \rangle$. Therefore a is semisimple.

(6) $\Rightarrow$ (1): Suppose that a is semisimple. Then $a \in \langle a \rangle^2$ and hence $a \in a^2S^1 = S^1a^2$. Therefore $a = a^2x = ya^2$ for some $x, y \in S^1$. Therefore a is left regular and right regular. By theorem2.8, a is completely regular.

Theorem 3.3

A semigroup S is a regular duo semigroup if and only if the relation

$$L \cap R = LRS \tag{1}$$

Holds for every left ideal L and every right ideal R of S.

Proof: Let S be a semigroup with property (1) for any left ideal L and any right ideal R of S. Then (1) implies

$$R = SRS \tag{2}$$

For any right ideal R of S, i.e. every right ideal R of S is two-sided.

Similarly, (1) implies

$L = LS^2$ (3)

for each left ideal L of S, that is each left ideal L of S is two-sided.

Therefore S is a duo semigroup. Next we show that S is regular.

For any (two-sided) ideal I of S (1) implies

$I = I^2S = IS^2 = SIS$ (4)

Hence we get

$I^2 = (SIS)(SIS) = SI,$ (5)

and

$I^2 = I(IS^2) = (IS^2)S = IS$ (6)

for every ideal I of S.

(5) and (6) imply

$IS = SI$ (7)

for any ideal I of S.

Finally (4) and (7) imply the relation

$I = ISI$

for each ideal I of S. This guarantees the regularity of S.

Conversely, let S be a regular duo semigroup. Then we have

$I_1 \cap I_2 = I_1 I_2$ (9)

for any couple of (two-sided) ideals of S. (9) implies

$I = IS = SI$ (10)

for any ideal I of S. (9) and (10) imply

$I_1 \cap I_2 = I_1 I_2 S$ (11)

for any couple of two-sided ideals of S. Therefore the relation (1) is true or every left ideal I and every right ideal I of S.

The proof of Theorem is completed.

We notice that the statement of the above Theorem remains true with associative ring instead of semigroup. The proof is analogous to that of Theorem.

4. Results on Duo Rings

Theorem 4.1

An associative ring S is a regular duo ring if and only if the relation (1) holds for each left ideal L and each right ideal R of S.

The proof of the following criterion is quite similar to that of the above Theorem.

Theorem4.2

A semigroup S is a regular duo semigroup if and only if the relation

$L \cap R = SLR$ (12)

holds for every left ideal L and every right ideal R of S.

Theorem 4.3

An associative ring S is a regular duo ring if and only if the relation (12) holds for every left ideal L and every right ideal R of S.

It may be remarked that we have some further ideal-theoretic identities any one of which characterizes the class of regular duo semigroups.

Theorem 4.4

A semigroup S is a semilattice of groups if and only if the relation

$$R \cap L = LR \quad (13)$$

holds for any left ideal L and right ideal R of S.

Definitions 4.5

A ring A is called a duo (or two-sided) ring if every one-sided (left or right) ideal of A is a two-sided ideal (see Hille [16]). An associative ring A is said to be regular if to every element a in A there exists an element x $\in$A such that axa=a (see yon Neumann [19]). Interesting characterizations of regular rings were obtained by Kovcs [17] and Luh [15].

Theorem 4.6

An associative ring A is a regular duo ring if and only if the relation (13) holds for any left ideal L and right ideal R of A.

Proof: Necessity. Let A be a regular duo ring. Then A satisfies the relation

$$R \cap L = RL \quad (14)$$

for every left ideal L and right ideal R of A. Now (14) implies (13) because each one-sided ideal of a duo ring is a two-sided ideal.

Sufficiency. Let A be an associative ring having the property (13) for any left ideal L and right ideal R of A. In case of R=A the relation (13) implies

$$A \cap L = LA, \quad (15)$$

hence every left ideal L of A is also a right ideal of A. Similarly, in case of L = A (13) implies that

$$R \cap A = AR, \quad (16)$$

whence the right ideal R of A is a two-sided ideal of A. Therefore

A is a duo ring. Finally (13) implies (14), which is equivalent with the regularity of A.

Theorem is completely proved.

5. Conclusions

Few results on duo semigroups with regular property are established. In addition to this, the regular property of duo semirings is also discussed with some fruitful results.

References

[1] A. Anjaneyulu, Structure and ideal theory of duo semigroups, Semigroup Forum, 22 (1981).
[2] A. Cherubini and A. Varisco, Permutable duo semigroups, Semigroup Forum, 28 (1984).
[3] A. Cherubini and A. Varisco, Semigroups whose proper subsemigroups are duo, Czechoslovak Math. J., 34 (1984).
[4] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Mathematical Surveys, No. 7, vol. 1 and 2, American Mathematical Society, Providence.
[5] O. Groek and L. Satko, A new notion in the theory of semigroups, Semigroup Forum, 20 (1980).
[6] K. M. Kapp, On bi-ideals and quasi-ideals in semigroups, Publ. Math. Debrecen, 16 (1969).
[7] Y. Kemprasit and C. Namnak, Generalized transformation semigroups whose bi-ideals and quasi-ideals coincide, Southeast Asian Bull. Math., 27 (2003).
[8] S. Lajos, On regular right duo semigroups, Elem. Math., 27 (1972).
[9] N. Kehayopulu, Right regular and right duo ordered semigroups, Math. Japon. 36(2) (1991), 201-206.

[10] S. Lajos" Note on semigroups, which are semilattices of groups. Proc. Japan Acad., 44, 805-806 (1968).
[11] S. Lajos: On semilattices of groups. Proc. Japan Acad., 48, 383-384 (1969).
[12] S. Lajos: On regular duo rings, Proc. Japan Acad., 48, 157-158 (1969).
[13] S. Lajos: Bi-ideals in Associative Rings. K. Marx University of conomics, Department of Mathematics. Budapest (1970).
[14] H. Clifford and G. B. Preston: The Algebraic Theory of Semigroups, Vol. 1 (2nd edition). Amer. Math. Soc., Providence R. I. (1964).
[15] J. Luh: A characterization of regular rings. Proc. Japan Acad., 39, 741-742 (1963).
[16] E. Hille: Functional analysis and semi-groups. Amer. Math. Soc., Coll. Publ. New York (1948).
[17] L. Kovcs" A note on regular rings. Publ. Math. Debrecen, 4, 465-468 (1955-56).
[18] J. Luh: A characterization of regular rings. Proc. Japan Acad., 39, 741-742 (1963).
[19] J. yon Neumann: On regular rings. Proc. Nat. Acad. Sci. U. S. A., 22,707-713 (19a6).
[20] http://en.wikipedia.org/