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On VIM-Pad`e Method for Boundary Layer Flow over an Exponentially Stretching Surface

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Abstract: *The problem of steady boundary layer flow over an exponentially stretching surface is studied. The governing partial differential equations are converted to yield a system of non-linear, coupled, ordinary differential equations by a similarity transformation group. These equations are analytically solved via VIM-Pad`e technique which is a combination of the Variational Iteration Method (VIM) and the Pad`e approximation for a benchmark testing of alternative numerical solutions. The physical parameters of interest (Prandtl number,* Pr *and stretching parameter,* k *) sensitively influence the local Nusselt number,* $-\theta'(0)$ *, thereby affecting the thermal boundary layer thickness. It is found that the rate of heat transfer from the fluid to the surface increases with increasing physical parameters. The VIM-Pad`e is implemented without requiring linearization, discretization, or perturbation. The results demonstrated reasonable degree of agreement when compared to other techniques and numerical methods reported in the literature. This suggests that the VIM with the enhancement of Pad'e; approximation is a very effective, convenient and quite accurate tool that requires further investigations by its application to viable technological and engineering problems that lead to nonlinear partial or differential equations.*

Keywords: *Boundary Layer Flow; Finite Difference Technique; Pad`e Approximation; Variational Iteration Method.*

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1. INTRODUCTION

Boundary layer flow is a subject of high interest in physics, fluid mechanics and engineering processes due to the associated viscosity effects on the bounding surface. There are quite a number of occurrences of boundary-layer flow and heat transfer of viscous fluids over a stretching surface in manufacturing processes, such as wire drawing, metal and polymer extrusion, drawing of copper wires, hot rolling, paper production, glass-fiber, glass blowing, electronic chips, crystal growing, and metal spinning [1, 2]. Magyari and Keller [3] investigated boundary layers on an exponentially stretching continuous temperature distribution. It was observed that the kinematics of stretching and the simultaneous heating or cooling during such processes have a decisive influence on the quality of the final products. For example, the knowledge of flow and heat transfer within a thin liquid film is crucial in understanding the coating process and design of exchangers and engineering equipments [4]. It is emphasized here that the subject of heat transfer is of fundamental importance in many branches of engineering such that it enables the study of various thermodynamic processes and their effects in equipment design (heat transfer enhancement), insulation properties, material selection, bio-heat transfer and many more [5]. The monitoring of extrusion stability of thin film layers is aimed at controlling the coating efficiently to maintain the surface quality of extrudes, and this is very important in textile and plastic industries. The problem of boundary layer on an exponentially stretching sheet has been extended to incorporate many other effects such as viscous dissipation [6], thermal radiation [7], magnetohydrodynamics [1] and many more. All these are to demonstrate the very many applications of the boundary layer flow on a continuous stretching sheet in industrial manufacturing processes. Just as there are many varieties of applications, different solution methods are advanced for the governing equations ranging from analytical to numerical, depending on the nature of the problem.

Most problems encountered in engineering and technological fields are governed by linear or nonlinear partial or ordinary differential equations. The solutions of these equations are not easily amenable to exact results. Therefore, the development of approximate techniques that are geared towards providing at least simple approximate analytical solutions for validating numerical results is apparently common and very much in order. The expected outcome of these methods is to reveal the characteristics or phenomenon under study. Of course, analytical and numerical investigations are complementary.

It is the objective of this paper, therefore, to construct an analytical result complemented by numerical solution to the problem of boundary layer flow over an exponentially stretching surface via VIM-Pad`e. The sections followed hereafter respectively are: the mathematical formulation of the problem, the construction of the solution via variational iteration method, the Pad`e approximation, discussion of results and general concluding remarks of the results of the previous sections.

2. MATHEMATCAL FORMULATION

Mohyud-Din et al. [8] investigated the problem of steady boundary layer flow on a continuous stretching surface, when the velocity and temperature of the surface varies exponentially with the distance along the sheet. The laminar boundary layer equations expressing the conservation of mass, momentum, and energy of the Newtonian fluids in the absence of free convection in the momentum equation are written as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{dy} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where x and y axes are taken along and perpendicular to the plate, respectively, and u and v are the velocity components parallel and normal to the plate, respectively, ρ is the fluid density, ν the kinematic viscosity, κ the thermal conductivity, c_p the specific heat and T the fluid temperature in the boundary layer. The appropriate accompanying boundary conditions are

$$\begin{aligned} &u = U_0 \exp\left(\frac{x}{L}\right), v = 0, T = T_\infty + T_0 \exp\left(\frac{kx}{2L}\right) \text{ for } y = 0, \\ &u = U_\infty, T = T_\infty \ as\ y \to \infty, \end{aligned} \tag{4}$$

where $L > 0$ is a reference length, and $U_0 > 0$, a reference velocity parameter of the stretching surface, and T_0 a reference temperature of the temperature distribution in the stretching surface and k the stretching parameter in the surface.

The continuity equation (1) could be satisfied by introducing a stream function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}. \tag{5}$$

The momentum and energy equations (2, 3) could be transformed into corresponding nonlinear ordinary differential equations via the similarity transformation group:

$$\begin{aligned}
u &= U_0 \exp\left(\frac{x}{L}\right) f'(\eta), \\
v &= -\left(\frac{\nu U_0}{2L}\right)^{\frac{1}{2}} \exp\left(\frac{kx}{2L}\right)\{f(\eta) + \eta\, f'(\eta)\}, \\
T &= T_\infty + T_0 \exp\left(\frac{k\,x}{2\,L}\right)\theta(\eta), \\
\eta &= \left(\frac{U_0}{2L}\right)^{\frac{1}{2}} y \exp\left(\frac{k\,x}{2\,L}\right),
\end{aligned} \tag{6}$$

where η is the similarity variable, $f(\eta)$ is the dimensionless stream function, $\theta(\eta)$ is the dimensionless temperature and primes denote differentiation with respect to η. Therefore, the transformed ordinary differential equations are:

$$f''' + f\,f'' - 2f'^2 = 0, \tag{7}$$

$$\theta'' + \Pr\left(f\theta' - kf'\theta\right) = 0, \tag{8}$$

where $\Pr = \dfrac{\rho\nu\, c_p}{\kappa}$ is the Prandtl number, which is the ratio of viscous to thermal diffusion. The transformed boundary conditions are:

$$\begin{aligned}
&f(0) = 0,\ f'(0) = 1,\ \theta(0) = 1, \\
&f'(\eta) \to 0, \theta(\eta) \to 0 \ \text{ as } \ \eta \to \infty.
\end{aligned} \tag{9}$$

From the physical point of view, the quantities of interest are the skin friction coefficient, $f''(0)$ and the local Nusselt number, $-\theta'(0)$, which represent the wall shear stress and the heat transfer rate at the surface, respectively. The momentum equation (7) is standing alone but is nonlinear and is coupled to the energy equation (8). The attempt to obtaining exact analytical solutions to these equations is elusive, but approximate analytical and numerical results are in order. The main thrust of this paper, therefore, is to construct approximate analytical solutions for the computation of the skin friction coefficient, $f''(0)$ and the local Nusselt number, $-\theta'(0)$ via VIM-Pad`e.

3. Variational Iteration Method (VIM) and Pad`e Approximant

The VIM is an effective approximate analytical method for solving various kinds of linear and nonlinear problems encountered in chemical, ecological, biological, and engineering applications [9, 10]. The method is a modified general Lagrange's multiplier method [11]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as an initial approximation or trial function. Then a more highly precise approximation at some special point could be obtained. This approximation converges rapidly to an accurate solution. To illustrate the basic concepts of the VIM, the following nonlinear differential equation is considered:

$$Lu + Nu = g(t), \tag{10}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is an inhomogeneous term. According to the VIM, a correction functional is constructed as follows:

$$u_{n+1}(t) = u(t) + \int_0^t \lambda(\xi)\left\{Lu_n(\xi) + N\,\overline{u_n(\xi)} - g(\xi)\right\}d\xi, \tag{11}$$

where $\lambda(\xi)$ is a general Lagrangian multiplier, which could be identified optimally via the variational theory, the subscript n denotes the nth -order approximation, $\overline{u_n}$ is considered as a restricted variation, that is, $\delta \overline{u_n(\xi)} = 0$.

Now constructing a correction functional to the momentum and energy equations (7, 8) respectively in the form:

$$f_{n+1}(\eta) = f(\eta) + \int_0^\eta \lambda_1(\xi)\left\{f_n'''(\xi) + \overline{f_n(\xi) f_n''(\xi)} - \overline{2 f_n(\xi)^2}\right\} d\xi, \tag{12}$$

$$\theta_{n+1}(\eta) = \theta(\eta) + \int_0^\eta \lambda_2(\xi)\left\{\theta_n''(\xi) + \Pr\left(\overline{f_n(\xi)\theta_n'(\xi)} - \overline{k\, f_n'(\xi)\theta_n(\xi)}\right)\right\} d\xi, \tag{13}$$

yield the following stationary conditions:

$$\begin{aligned} &1 + \lambda_1''(\xi) = 0\,|_{\xi=\eta},\ \lambda_1(\xi) = 0\,|_{\xi=\eta}, \lambda_1'(\xi) = 0\,|_{\xi=\eta}, \lambda_1'''(\xi) = 0, \\ &1 - \lambda_2'(\xi) = 0\,|_{\xi=\eta},\ \lambda_2(\xi) = 0\,|_{\xi=\eta}, \lambda_2''(\xi) = 0. \end{aligned} \tag{14}$$

The Lagrange multipliers, therefore, are identified respectively as

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}(\xi - \eta)^2, \\ \lambda_2 &= (\xi - \eta). \end{aligned} \tag{15}$$

Now, let $f''(0) = \alpha$ and $\theta'(0) = \beta$. Therefore, the equations for the respective iterative solutions for momentum and energy now become

$$f_{n+1}(\eta) = f(\eta) - \frac{1}{2}\int_0^\eta (\xi - \eta)^2 \left\{f_n'''(\xi) + f_n(\xi) f_n''(\xi) - 2 f_n(\xi)^2\right\} d\xi, \tag{16}$$

$$\theta_{n+1}(\eta) = \theta(\eta) + \int_0^\eta (\xi - \eta)\{\theta_n''(\xi) + \Pr\left(f_n(\xi)\theta_n'(\xi) - k\, f_n'(\xi)\theta_n(\xi)\right)\} d\xi. \tag{17}$$

Consequently, the first three iterations of the momentum and energy equations are respectively:

$$\begin{aligned} f_0 &= \eta + \frac{1}{2}\alpha\eta^2, \\ f_1 &= \eta + \frac{1}{2}\alpha\eta^2 + \frac{1}{3}\eta^3 + \frac{1}{8}\alpha\eta^4 + \frac{1}{40}\alpha^2\eta^5, \\ f_3 &= \eta + \frac{1}{2}\alpha\eta^2 + \frac{1}{3}\eta^3 + \frac{1}{8}\alpha\eta^4 + \left(\frac{1}{30} + \frac{1}{40}\alpha^2\right)\eta^5 + \frac{1}{96}\alpha\eta^6 + \left(\frac{2}{315} + \frac{3}{560}\alpha^2\right)\eta^7 \\ &+ \left(\frac{5}{1344}\alpha + \frac{3}{4480}\alpha^3\right)\eta^8 + \frac{143}{12060}\alpha^2\eta^6 + \frac{1}{4800}\alpha^3\eta^{10} + \frac{1}{52800}\alpha^4\eta^{11}, \\ &\vdots, \\ \theta_0 &= 1 + \beta\eta, \\ \theta_1 &= 1 + \beta\eta + \frac{1}{2}k \Pr \eta^2 - \frac{1}{6}\Pr(\beta - k\beta - k\alpha)\eta^3 - \frac{1}{12}\Pr(\alpha\beta - k\alpha\beta)\eta^4, \end{aligned} \tag{18}$$

$$\theta_2 = 1+\beta\eta+\frac{1}{2}k\Pr\eta^2-\frac{1}{6}\Pr(\beta-k\beta-k\alpha)\eta^3-\frac{1}{12}\Pr\begin{pmatrix}\alpha\beta+k\Pr\\ -k-k\alpha\beta-\frac{1}{2}k^2\Pr\end{pmatrix}\eta^4$$

$$-\frac{1}{20}\Pr\begin{pmatrix}\frac{1}{6}k\Pr(\beta-k\beta-k\alpha)-\frac{1}{2}k^2\alpha\Pr-k\beta-\frac{1}{2}k\alpha-\frac{1}{2}\Pr(\beta-k\beta-k\alpha)\\ +\frac{1}{2}k\alpha\Pr+\frac{1}{3}\beta\end{pmatrix}\eta^5$$

$$-\frac{1}{30}\Pr\begin{pmatrix}-\frac{1}{3}\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)-\frac{1}{4}\alpha\Pr(\beta-k\beta-k\alpha)+\frac{1}{3}k\Pr+\frac{1}{8}\alpha\beta\\ +\frac{1}{12}k\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)+\frac{1}{6}k\alpha\Pr(\beta-k\beta-k\alpha)-\frac{1}{2}k^2\Pr-\frac{1}{8}k\alpha^2\end{pmatrix}\eta^6$$

$$-\frac{1}{42}\Pr\begin{pmatrix}\frac{1}{12}k\alpha\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)+\frac{1}{6}k\Pr(\beta-k\beta-k\alpha)-\frac{1}{4}k^2\alpha\Pr-\frac{1}{8}k\alpha^2\beta\\ -\frac{1}{6}\alpha\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)-\frac{1}{6}\Pr(\beta-k\beta-k\alpha)+\frac{1}{8}k\alpha\Pr+\frac{1}{40}\alpha^2\beta\end{pmatrix}\eta^7$$

$$-\frac{1}{56}\Pr\begin{pmatrix}\frac{1}{12}k\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)+\frac{1}{12}k\alpha\Pr(\beta-k\beta-k\alpha)-\frac{1}{16}k^2\alpha^2\Pr\\ -\frac{1}{9}\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)-\frac{1}{16}\alpha\Pr(\beta-k\beta-k\alpha)+\frac{1}{40}k\alpha^2\Pr\end{pmatrix}\eta^8$$

$$-\frac{1}{72}\Pr\begin{pmatrix}\frac{1}{24}k\alpha\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)+\frac{1}{48}k\alpha^2\Pr(\beta-k\beta-k\alpha)-\frac{1}{24}\alpha\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)\\ -\frac{1}{80}\alpha^2\Pr(\beta-k\beta-k\alpha)\end{pmatrix}\eta^9$$

$$-\frac{1}{90}\Pr\left(-\frac{1}{120}\alpha^2\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)+\frac{1}{96}k\alpha^2\Pr\left(\frac{1}{2}\alpha\beta-k\alpha\beta\right)\right)\eta^{10}, \qquad (19)$$

$\vdots$

Thus, the solutions of the momentum and energy equations are respectively approximated by the series solution:

$$f(\eta) = f_2, \qquad (20)$$

$$\theta(\eta) = \theta_2. \qquad (21)$$

For want of more accuracy, as many iterations as possible may be computed, but with much lengthier expressions. One advantage of VIM is that in some physical problems only few iterations may be computed and it converges to the required result. Considerations of the convergence of VIM are elsewhere reported in literature [12]. The main task now is to obtain numerical values of α and β, respectively, using the asymptotic condition ($f'(\eta)\to 0, \theta(\eta)\to 0$ as $\eta\to\infty$). It is pertinent to note that this condition could not be applied directly to the results (20, 21). It is known that this goal could be achieved via pad`e approximant, which has the advantage of converting the series solutions to rational functions [13 - 16].

Pad`e approximations are derived by expanding a function as a rational polynomial of two power series. These approximations are usually called Pad`e approximants, which are usually superior to Taylor expansions, especially when the functions contain poles. The rational approximation of a truncated series solution $f(x)$, say, of order at least $(N+M)$ on $[a,b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degree N and M, respectively. The power series $f(x)=\sum_{i=0}^{n} c_i x^i$ is denoted by the quotient $[N/M]=\frac{P_N(x)}{Q_M(x)}$ for $a \le x \le b$, where $Q_M(x) \ne 0$. This implies that without loss of generality, assume the normalization condition $Q_M(0)=1$. Further, P_N and Q_M have no common factors. This requires that $f(x)$ equals the $[N/M]$ approximant through $N+M+1$ terms. The pad`e method is to make the maximum error as small as possible. Built-in utilities of symbolic computing packages such as Maple or Mathematica could be used to obtain the Pad`e approximants of different orders $[N/M]$ efficiently. It is known that the diagonal approximants $[N/M]$ like $[2/2],[3/3],[4/4],[5/5]$ or $[6/6]$ have been confirmed to be most accurate approximants [8]. In this paper, the diagonal approximants $[2/2],[3/3]$ and $[4/4]$ are used for the computations.

4. DISCUSSION OF RESULTS

The nonlinear ordinary differential equations (7, 8) subjected to (9) have been solved using the VIM-Pad`e. Using the Pad`e $[2/2],[3/3]$ and $[4/4]$, the results (20, 21) uniquely determine the wall shear stress, $f''(0)$ as -1.290994, -1.276792 and -1.290957, respectively, while numerical solution gives -1.281816, which corresponds to the result obtained by Bidin and Nazer [17]. Note that the numerical computation uses a finite difference technique with Richardson extrapolation, which is unconditionally stable [18] and was implemented in MAPLE. To select $\eta \to \infty$ for the numerical solution, some initial guess value is used to solve the problem with some particular set of parameters. The solution process is repeated with another larger value of $\eta \to \infty$ until two successive values of the results differ only after desired digit(s) signifying the limit of the asymptotic boundary. Therefore, in the computations for the value of β use is made of the value of $\alpha = 1.276792$, which is obtained from the Pad`e $[3/3]$.

Table 1 displays the heat flux, $\theta'(0)$ as computed from the result (21) by the use of Pad`e $[3/3]$ and are compared with those obtained by Magyari and Keller [3] utilizing shooting method (SM) and approximation formula (AF), Mohyud-Din et al. (2010) using Homotopy Perturbation Method, and numerical results obtained for several values of Pr for $k=3$. It is observed that there is high correlation between VIM-Pad`e and those reported in Magyari and Keller (1999), Mohyud-Din et al. (2010) and the numerical results. This demonstrates the effectiveness of the VIM-Pad`e.

Table 1. *Heat flux at the surface, $\theta'(0)$ for several values of* Pr *for* $k=3$

Pr	Magyari and Keller (1999)		Mohyud-Din et al. (2010) [3/3]	Present Results	
	SM	AF		Numerical,	VIM-Pad`e
0.5	-1.008405,	-0.990315	-1.0014	-1.009281,	-1.007413
1	-1.560294,	-1.550413	-1.5631	-1.560295,	-1.544487
3	-2.938535,	-2.939387	-2.9127	-2.938531,	-2.960783
5	-3.886555,	-3.890628	-3.8357	-3.886552,	-3.902319
8	-5.000460,	-5.006760	-4.9197	-5.000461,	-5.002728

10	-5.628190, -5.635360	-5.5311	-5.628194, -5.621850

It is also observed from the Table 1 that as the Prandtl number increases, the heat flux $\theta'(0)$ decreases. Physically, since the Prandtl number is a measure of the ratio of viscous to thermal diffusion, increase in the Prandtl number indicates a decrease in the thermal diffusion, thereby reducing the thermal boundary layer thickness, and hence increases the rate of heat transfer at the surface.

Table 2 displays the heat flux determined by the VIM-Pad`e $[2/2]$ and $[3/3]$ for several values of $k$ for $\Pr = 1$ and $\Pr = 10$, where the VIM-Pad`e $[2/2]$ and $[3/3]$ are shown respectively in brackets in the present results column. It is observed that for a given Prandtl number, the heat flux decreases with increasing stretching parameter. The increasing negative values of the wall temperature gradient are indicative of the physical fact that the heat flows from the ambient fluid to the plate surface.

Table 2. *Heat flux determined using the VIM-Pad`e* $[2/2]$ *and* $[3/3]$ *for several values of* k *for* $\Pr = 1$ *and* $\Pr = 10$ *, respectively*

k	**Mohyud-Din et al. (2010)** $\Pr = 1$		Present Results $\Pr = 10$ **Numerical (VIM-Pad`e)**	**Mohyud-Din et al. (2010)** $\Pr = 10$		Present Results $\Pr = 1$ **Numerical (VIM-Pad`e)**
	$[2/2]$	$[3/3]$		$[2/2]$	$[3/3]$	
5	-2.0250,	-2.0261	-2.026967 (-2.001745) (-2.059560)	-7.4219,	-7.0665	-7.111832 (-7.423252) (-6.974107)
7	-2.4593,	-2.4164	-2.418099 (-2.460770) (-2.387348)	-8.6742,	-8.3114	-8.349557 (-8.675419) (-8.137707)
8	-2.6404,	-2.5924	-2.594451 (-2.641745) (-2.561167)	-9.2422,	-8.8699	-8.907038 (-9.243458) (-8.694616)
9	-2.8112,	-2.5924	-2.760860 (-2.812579) (-2.725145)	-9.7795,	-9.3962	-9.432952 (-9.780732) (-9.218065)
10	-2.9735,	-2.9162	-2.918817 (-2.974798) (-2.880761)	-10.2904,	-9.8953	-9.932092 (-10.29161) (-9.713659)

5. Concluding Remarks

The problem of the boundary layer flow over an exponentially stretching surface has been examined. Analytical solutions of the flow variables are presented via VIM-Pad`e technique. It is generally observed that the physical parameters (that is, the Prandtl number, $\Pr$ and stretching parameter, k) entering the problem significantly influence the flow variables. The results demonstrated reasonable degree of agreement when compared to other techniques and numerical methods reported in the literature. This suggests that the VIM with the enhancement of Pad`e approximation is a very effective, convenient and quite accurate tool that requires further investigations by its application to viable technological and engineering problems that lead to nonlinear partial or differential equations. The main conclusions are the following:

1. The heat flux decreases with increasing Prandtl number for a given stretching parameter.

2. The thermal boundary layer thickness decreases with increasing Prandtl number, implying a slow rate of thermal diffusion. Therefore, higher Prandtl number leads to faster cooling of the surface.

3. For a given Prandtl number, the heat flux decreases with increasing stretching parameter.

4. The negative values of the wall temperature gradient are indicative of the physical fact that the heat flows from the plate surface to the ambient fluid.

5. The VIM-Pad`e technique vis-a-vis the analytical solution can minimize time wasting and complicated calculations of numerical methods and many successive approximations.

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