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Total Domination on Generalized Petersen Graphs

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Abstract: *A total dominating set of a graph G is a set of the vertex set V of G such that every vertex of G is adjacent to a vertex in S. In this paper, we have developed an algorithm to find the minimal total dominating set of the generalized Petersen graphs $P(n,k)$ when $n \geq 2k+1, k = 1,2$.*

Keywords: *neighborhood, domination, total domination and generalized Petersen graphs.*

1. Introduction

Cockayne et al., [1] have introduced the concept of total domination set in graphs and this field is under the study of many researches. Teresa et al., [2] have given the comprehensive treatment of theoretical, algorithmic and application aspects of domination in graphs in detail and a survey of several advanced topics in dominations are also given.

In any real world situation which can be modeled by a graph and where domination is of interest, the particular locations commanding high domination values-strategic high grounds are obviously important.

Definition 1.1 The open neighborhood of a vertex $v \in V(G)$ is denoted by $N(v)$ and is defined as

$N(v) = \{u \in V(G) | uv \in E(G)\}$

The closed neighborhood of a vertex $v \in V(G)$ is denoted by $N[v]$ and is defined as

$N[v] = N(v) \cup \{v\}$

Definition 1.2 The set $S \subset V$ of vertices in a graph $G = (V,E)$ is a dominating set if every vertex $v \in V$ is an element of S or adjacent to an element of S.

Definition 1.3 A dominating set S of G is a total dominating set of G if every vertex of G is adjacent to a vertex in S and we represent it as $TD-set$.

Thus, a set $S \subseteq V$ is a $TD-set$ in G if $N(S) = V$.

Definition 1.4 The total domination number of G, denoted by $\gamma_t(G)$, is the cardinality of the minimal $TD-set$ of G

Definition 1.5 Let n,k be positive integers such that $n \geq 3$ and $1 \leq k \leq \left[\frac{n}{2}\right]$. The generalized Petersen graph $P_{n,k}$ is the graph whose vertex set is $\{a_i, b_i : 1 \leq i \leq n\}$ and whose edge set is $\left\{\{a_i,b_i\},\{a_i,a_{i+1}\},\{b_i,b_{i+k}\}: 1 \leq i \leq n\right\}$ where $a_{n+c} = a_c$ and $b_{n+c} = b_c$ for every $c \geq 1$.

Throughout this paper, we take the outer vertices as $u_1, u_2, \ldots u_n$ and the inner vertices as $v_1, v_2, \ldots v_n$ for $P(n,k)$.

2. Total Dominating Set of the Generalized Petersen Graphs $P(n,1)$

Theorem 2.1 The minimal total dominating set for the generalized Petersen graphs $P(n,1)$ with $n \geq 3$ except $n = 7$ is given by

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \end{cases}$$

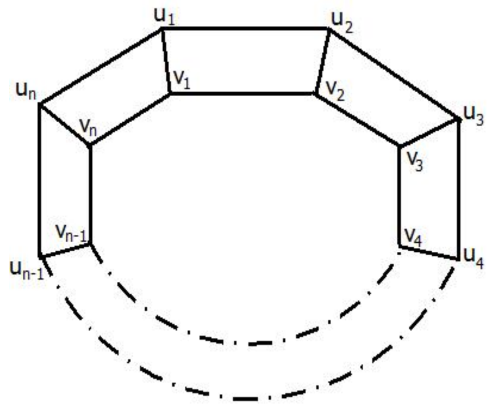

Figure 1. Generalized Petersen graph P(n,1)

Proof. Let $n \ge 3$ and $n \ne 7$. The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition i) ; and the vertex v_{1+3i} dominates the vertices v_{3i}, v_{3i+2} and u_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition i). For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_2, v_n and u_1. As i ranges from 0 to $\left\lceil \frac{n}{3} \right\rceil$, the minimal total dominating set thus obtained is as follows

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \end{cases}$$

Example 2.2 Consider the generalized Petersen graph $P(6,1)$. Let $u_1, u_2, \dots u_6$ be the outer vertices and $v_1, v_2, \dots v_6$ be the corresponding inner vertices.

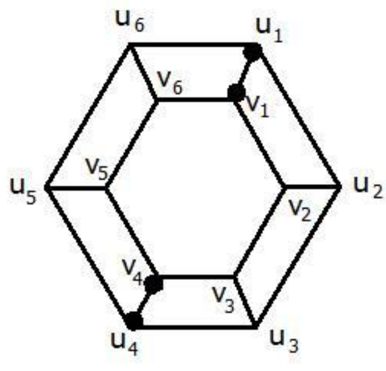

Figure 2. Generalized Petersen graph P(6,1)

By applying theorem 2.1, the minimal total dominating set of $P(6,1)$ is $\{u_1, u_4, v_1, v_4\}$.

Remark 2.3 Consider the generalized Petersen graph $P(7,1)$ when $n = 7$. Let $u_1, u_2, \dots u_7$ be the outer vertices and $v_1, v_2, \dots v_7$ be the corresponding inner vertices.

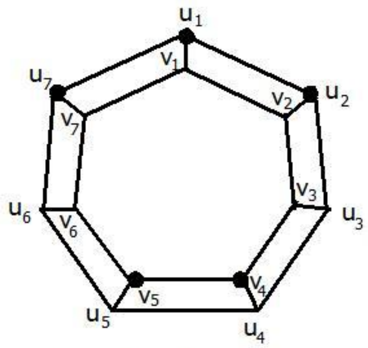

Figure 3. Generalized Petersen graph P(7,1)

The vertex u_1 dominates the vertices u_2, u_7 and v_1; the vertex u_2 dominates the vertices u_1, u_3 and v_2; the vertex u_7 dominates the vertices u_1, u_6 and v_7; the vertex v_4 dominates the vertices v_3, v_5 and u_4; and the vertex v_5 dominates the vertices v_4, v_6 and u_5. Thus a set of vertices $\{u_1, u_2, u_7, v_4, v_5\}$ dominates every vertex of $P(7,1)$. Thus the minimal total dominating set is $\{u_1, u_2, u_7, v_4, v_5\}$.

3. TOTAL DOMINATING SET OF THE GENERALIZED PETERSEN GRAPHS $\mathbf{P(n, 2)}$

Theorem 3.1 The minimal total dominating set for the generalized Petersen graph $P(n, 2)$ is given by

(i) For n even, $n > 8$there are two cases :

(a) $n \not\equiv 2(mod6)$

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \end{cases}$$

(b) $n \equiv 2(mod6)$

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ & v_{n-2} \end{cases}$$

(ii)For n odd, $n > 5$there are two cases :

(a) $n \equiv 0(mod3)$and $n \equiv 1(mod3)$

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \end{cases}$$

(b) $n \equiv 2(mod3)$

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ & v_{n-2} \end{cases}$$

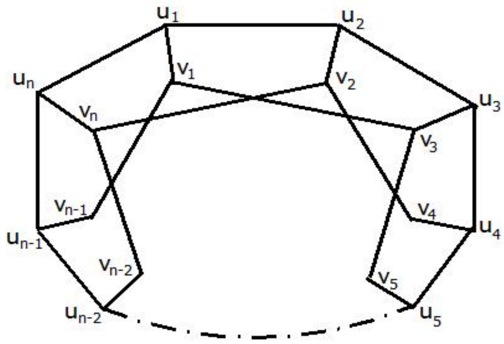

Figure 4. Generalized Petersen graph P(n,2)

Proof. (i) Let n be even and $n > 8$. There are two cases:

*Case (a):*Let $n \not\equiv 2(mod6)$. The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition *i*) ; and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition *i*). For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1. We get the TD-set of $P(n, 2)$ for all values of *i*, $0 \le i < \left\lceil \frac{n}{3} \right\rceil$ as

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \end{cases}$$

*Case (b):*Let $n \equiv 2(mod6)$. The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$; and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition *i*). For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1; and also the vertex v_{n-2} dominates the vertices v_{n-4}, v_n and u_{n-2}. We get the TD-set of $P(n, 2)$ for all values of *i*, $0 \le i < \left\lceil \frac{n}{3} \right\rceil$ as

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ & v_{n-2} \end{cases}$$

(ii) Let n be odd and $n > 5$. There are two cases :

Case (a): Let $n \equiv 0(mod3)$and $n \equiv 1(mod3)$. The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition *i*) ; and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition *i*). For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1. We get the TD-set of $P(n, 2)$ for all values of *i*, $0 \le i < \left\lceil \frac{n}{3} \right\rceil$ as

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \end{cases}$$

*Case (b):*Let $n \equiv 2(mod3)$. The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$; and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \le i < \left\lceil \frac{n}{3} \right\rceil$ (modulo addition *i*). For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1; and also the vertex v_{n-2} dominates the vertices v_{n-4}, v_n and u_{n-2}. We get the TD-set of $P(n, 2)$ for all values of *i*, $0 \le i < \left\lceil \frac{n}{3} \right\rceil$ as

$$TD = \begin{cases} u_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ v_{1+3i}, & 0 \le i < \left\lceil \frac{n}{3} \right\rceil \\ & v_{n-2} \end{cases}$$

Remark 3.2 The values 5,6 and 8 of$n$ are not included in the above theorem. Here we have given separately the TD-set of $P(5,2)$, $P(6,2)$ and $P(8,2)$.

1. Consider the generalized Petersen graph $P(5,2)$ given in fig-5. Let $u_1, u_2, \dots u_5$ be the outer vertices and $v_1, v_2, \dots v_5$ be the corresponding inner vertices. The vertex u_1 dominates the vertices u_2, u_5 and v_1; the vertex v_1 dominates the vertices v_3, v_4 and u_1; the vertex v_3 dominates the vertices v_1, v_5 and u_3; and the vertex v_4 dominates the vertices v_1, v_2 and u_4. Thus the set of vertices $\{u_1, v_1, v_3, v_4\}$ dominates every vertex of $P(5,2)$. Thus the minimal total dominating set is$\{u_1, v_1, v_3, v_4\}$.

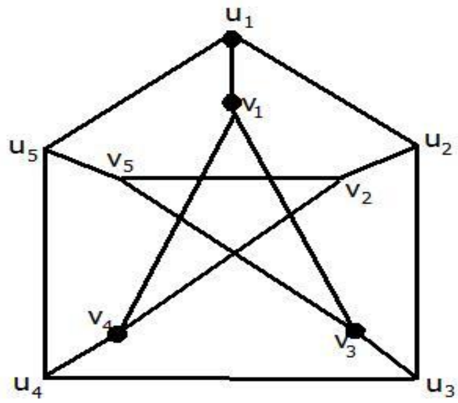

Figure 5. Generalized Petersen graph P(5,2)

2. Consider the generalized Petersen graph $P(6,2)$ given in fig-6. Let $u_1, u_2, \dots u_6$ be the outer vertices and $v_1, v_2, \dots v_6$ be the corresponding inner vertices. The vertex u_1 dominates the vertices u_2, u_6 and v_1; the vertex u_4 dominates the vertices u_3, u_5 and v_4; the vertex v_1 dominates the vertices v_3, v_5 and u_1; and the vertex v_4 dominates the vertices v_2, v_6 and u_4. Thus the set of vertices $\{u_1, u_4, v_1, v_4\}$ dominates every vertex of $P(6,2)$. Thus the minimal total dominating set is$\{u_1, u_4, v_1, v_4\}$.

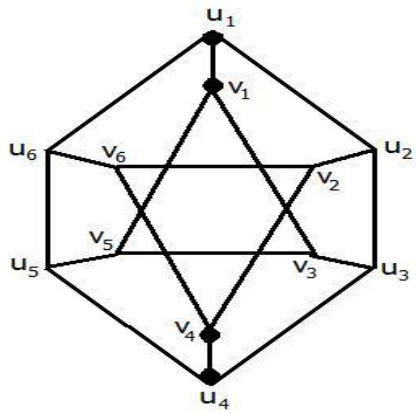

Figure 6. Generalized Petersen graph P(6,2)

3. Consider the generalized Petersen graph $P(8,2)$given in fig-7. Let $u_1, u_2, \dots u_8$ be the outer vertices and $v_1, v_2, \dots v_8$ be the corresponding inner vertices. The vertex u_1 dominates the vertices u_2, u_8 and v_1; the vertex u_4 dominates the vertices u_3, u_5 and v_4; the vertex v_1 dominates the vertices v_3, v_7 and u_1;the vertex v_4 dominates the vertices v_2, v_6 and u_4; the vertex v_6 dominates the vertices v_4, v_8 and u_6 and the vertex v_7 dominates the vertices v_1, v_5 and u_7. Thus the set of vertices $\{u_1, u_4, v_1, v_4, v_6, v_7\}$ dominates every vertex of $P(8,2)$. Thus the minimal total dominating set is$\{u_1, u_4, v_1, v_4, v_6, v_7\}$.

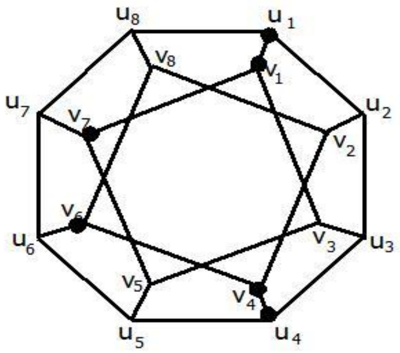

Figure 7. Generalized Petersen graph P(8,2)

4. In the above theorem 3.1, we note that the TD-set of the cases a(i) and a(ii) are same and for the cases b(i) and b(ii) also the TD-sets are same.

Example 3.3 Consider the generalized Petersen graph $P(10,2)$ to illustrate the theorem 3.1. Let $u_1, u_2, \dots u_{10}$ be the outer vertices and $v_1, v_2, \dots v_{10}$ be the corresponding inner vertices. Here $n = 10$ By applying theorem 3.1(case a(i)), the minimal total dominating set of $P(10,2)$ is $\{u_1, u_4, u_7, u_{10}, v_1, v_4, v_7, v_{10}\}$.

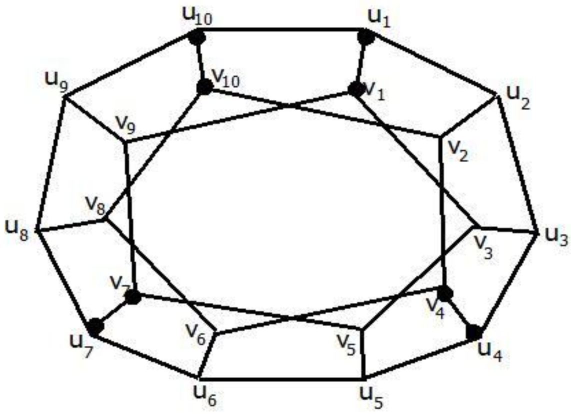

Figure 8. Generalized Petersen graph P(10,2)

4. CONCLUSION

In this paper we have found the minimal total dominating set of the generalized Petersen graphs $P(n, k)$ when $n \geq 2k + 1, k = 1,2$.

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