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On $|N, q_n, r_n|$- Summability of Jacobi Series

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Abstract: *In this paper we have established a theorem on* $|N, q_n, r_n|$ -summability *of Jocabi series, which gives some new interesting results and generalizes some previous known results.*

Keywords: $|N, q_n, r_n|$ *-summability method and Jacobi series.*

Mathematical classification: *40D25, 40E05, 40F05 & 40C10.*

1. Introduction

Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$ such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x)dx \tag{1.1}$$

Is exist in the sense of Lebesgue for $\alpha > -1$ and $\beta > -1$. The Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{1.2}$$

Where

$$a_n = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha+\beta)}(x) f(x)dx$$

If

$$b_n = \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \tag{1.3}$$

Then

$$a_n = b_n \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) f(x)dx \tag{1.4}$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials defined by the generating function

$$2^{\alpha+\beta}(1-2xt+t^2)^{-1/2}[1-t+(1-2xt+t^2)^{1/2}]^{-\alpha}$$

$$\times[1+t+(1-2xt+t^2)^{1/2}]^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n \tag{1.5}$$

Let us write

$$F(\phi) = \{f(\cos\phi) - A\}(\sin\phi/2)^{2\alpha+1}(\cos\phi/2)^{2\beta+1}$$

where A being a constant.

Let $\{s_n\}$ be the sequence of partial sums of an infinite series Σa_n. Let $\{r_n\}$ and $\{q_n\}$ be any two sequences of positive real constants with R_n and Q_n as their n-th partial sums respectively and let

$$(q*r)_n = \sum_{k=0}^{n} q_{n-k} r_k = \sum_{k=0}^{n} q_k r_{n-k} \text{ tends to infinity as } n \to \infty . \tag{1.6}$$

If the sequence to sequence transformation is defined by (Borwein [1])

$$t_n^{q,r} = \frac{1}{(q*r)_n} \sum_{k=0}^{n} q_{n-k} r_k s_k \tag{1.7}$$

If

$$t_n^{q,r} \to s \quad \text{as } n \to \infty$$

then the sequence of partial sums $\{s_n\}$ or infinite series Σa_n is said to be summable $|N, q_n, r_n|$ to s.

2. KNOWN RESULTS

Dealing with Nörlund summability of Jacobi series Pandey [7] has established the following theorem.

Theorem 2.1

Let $\alpha > -\frac{1}{2}, \; \beta - \alpha > -1, \; \beta + \alpha \geq -1$. Suppose that

$$\sum_{k=2}^{n} \frac{Q_n}{k^{\alpha+(t/2)} \log k} = O\left(\frac{Q_n}{n^{\alpha+(1/2)}}\right), \text{ as } n \to \infty \quad . \tag{2.1}$$

Also suppose that

$$\int_{1-t}^{1} |f(u) - A|\, du = O\left(\frac{t}{\log(\frac{1}{t})}\right), t \to O \tag{2.2}$$

and that the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} |f(x)|\, dx < \infty \tag{2.3}$$

is satisfied, where b is fixed then the series (1.2) is summable $|N, q_n|$ at the point $x = +1$ to the sum A.

3. MAIN RESULTS

The object of this paper is to generalize the Theorem 2.1 to a more general class on $|N, q_n, r_n|$-summability of the Jacobi series. .

Theorem 3.1

Let (N, q_n, r_n) be a summability method defined by a non-negative real constants sequences $\{q_n\}$ and $\{r_n\}$ and let $\alpha > -\frac{1}{2}, \beta - \alpha > 1, \beta + \alpha \geq -1$ such that

$$\sum_{k=2}^{n} \frac{(q*r)_k}{k^{\alpha+(1/2)} \log k} = O\left(\frac{(q*r)_n}{n^{\alpha+(1/2)}}\right) \text{ as } n \to \infty \tag{3.1}$$

Also suppose that

$$\int_{1-t}^{1} | f(u) - A | \, du = O\left(\frac{t}{\log(1/t)}\right), \text{ as } t \to O \tag{3.2}$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \, | f(x) | \, dx < \infty \tag{3.3}$$

are satisfied where b is fixed then the series (1.2) is summable $| N, q_n, r_n |$ at $x = +1$ to the sum A.

4. Lemmas

We have required the following lemmas to prove the theorem:

Lemma 4.1

Szego [10] for $\alpha > -1, \beta > -1$

$$P_n^{(\alpha,\beta)}(\cos\phi) = \begin{cases} O(n^{\alpha}), \text{ when } 0 \le \phi \le 1/n \\ O(n^{\beta}), \text{ when } \pi - \frac{1}{n} \le \phi \le \pi \\ \frac{1}{(n\pi)^{1/2}} (\sin\frac{\phi}{2})^{-(2\alpha+1)/2} (\cos\frac{\phi}{2})^{-(2\beta+1)/2} [\cos\left\{\frac{(2n+\alpha+\beta+1)}{2}\phi - (2\alpha+1)\frac{\pi}{4}\right\} + \frac{O(1)}{n\sin\phi} \\ \text{when } \frac{1}{n} \le \phi \le \pi - \frac{1}{n} \end{cases}$$

Lemma 4.2

let $\alpha > -\frac{1}{2}, \beta > -1$ and also let

$$N_n(\phi) = \frac{1}{(q*r)_n} (2)^{\alpha+\beta+1} \sum_{k=0}^{n} q_k r_{n-k} \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)$$

Where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \sim \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} n^{\alpha+1}$$

Then

For $0 \le \phi \le \frac{1}{n}$

$$| N_n(\phi) | = O(n^{2\alpha+2}) \tag{4.2}$$

For $\frac{1}{n} \le \phi \le \pi - \frac{1}{n}$

$$| N_n(\phi) | = O\left\{ \frac{1}{(q*r)_n} \frac{(n^{(2\alpha+1)/2} (q*r)_{(1/\phi)}}{(\sin\phi)^{(2\alpha+3)/2} (\cos\phi)^{(2\beta+1)/2}} \right\} \tag{4.3}$$

$$+ O\left(\frac{n^{(2\alpha-1)/2}}{(\sin\frac{\phi}{2})^{(2\alpha+\phi)/2} (\cos\frac{\phi}{2})^{(2\beta+3)/2}} \right)$$

For

$\pi - \frac{1}{n} \le \phi \le \pi$

$$| N_n(\phi) | = O(n^{\alpha+\beta+1}) \tag{4.4}$$

Proof:

For $\alpha > -\frac{1}{2}$ and $\beta > -1$ and $\{q_n\}$ and $\{r_n\}$ satisfy the conditions of theorem, using Lemma 4.1 for $0 \le \phi \le \frac{1}{n}$ then condition (4.2) is satisfied. For the estimation of (4.3) we use the Lemma (4.1) and Lemma (4.3) for $\pi - \frac{1}{n} \le \phi < \pi$.

For $\frac{1}{n} \le \phi \le \pi - \frac{1}{n}$ we have

$$N_n(\phi) = \frac{O(1)}{(q*r)_n} \sum_{k=1}^{n-1} q_k r_{n-k} (n-k)^{(2\alpha+1)/2} \ (\sin\frac{\phi}{2})^{-(2\alpha+1)/2} (\cos\frac{\phi}{2})^{-(2\beta+1)/2}$$

$$\times \left(\cos\{(n-k+\rho)\phi - \gamma\} + \frac{O(1)}{(n-k)\sin\phi} \right)$$

Since for fixed $n, (r_{n-k})$ is non-increasing we can deal with the first term of the right by using the second mean value theorem and apply the result of Lemma (4.3) and the required estimate follows.

Lemma 4.3:*(Khare [5]) If* $\{q_n\}$ *is a non-negative, non increasing and* $\{r_n\}$ *is a non-negative, non-decreasing* sequence, then

$$\sum_{k=0}^{n-1} q_k r_{n-k} (n-k)^{(2\alpha-1)/2} = O\left((q*r)_n n^{(2\alpha-1)/2}\right)$$

Lemma 4.4: *The assumption (3.1) implies that*

$$n^{\alpha+(\frac{1}{2})} = O\{q*r)_n\} \tag{4.5}$$

$$\text{where} \quad \alpha < \frac{1}{2} \tag{4.6}$$

Proof: The expression on the left of (3.1) is increasing and hence greater than equal to a positive constant. Hence (3.1) implies that, for some positive constant c

$$(q*r)_n > cn^{(2\alpha+\frac{1}{2})}$$

On substituting this, we see that the expression on the left of (3.1) tends to ∞ as $n \to \infty$ and (4.5) follows.

Since q_n and r_n are positive non increasing, $(q*r)_n = O(n)$ and (4.6) therefore follows by (4.5).

Lemma 4.5: (Pandey [7]) condition (3.2) is equivalent to

$$F_1(t) = \int_0^t | F(\phi) | d\phi = O\left(\frac{t^{2\alpha+2}}{\log(1/t)}\right) \text{ as } t \to 0 \tag{4.7}$$

Where

$$F(\phi) = [f(\cos\phi) - A](\sin\frac{\phi}{2})^{2\alpha+1}(\cos\phi/2)^{2\beta+1}.$$

Lemma 4.6: Let $\beta - \alpha > -1$. The antipole condition

$$\int_{-1}^{b}(1+x)^{(\beta-\alpha-1)/2}\,|f(x)|\,dx < \infty$$

Is equivalent to

$$\int_{-1}^{b}(1+x)^{(\beta-\alpha-1)/2}\,|f(x)-A|\,dx < \infty$$

Further

$$\int_{a}^{\pi}(\cos\tfrac{\phi}{2})^{-\alpha-\beta-1}\,|f(\phi)|\,d\phi < \infty \tag{4.8}$$

5. Proof of the Theorem

The n-th partial sum of the series (1.2), at the point $x = +1$ is given by Obrechkoff [6].

$$S_n(1) = 2^{\alpha+\beta+1}\int_0^{\pi}(\sin\tfrac{\phi}{2})^{2\alpha}(\cos\tfrac{\phi}{2})^{2\beta} f(\cos\phi)s_n(1,\cos\phi)\sin\phi\,d\phi \tag{5.1}$$

Where $S_n(1,\cos\phi)$ denote the n-th partial sum of the series

$$\sum_m \frac{P_m^{(\alpha,\beta)}(1)P_m^{(\alpha,\beta)}(\cos\phi)}{b_m}$$

Rao [9] has been shown that

$$S_n(1,\cos\phi) = \lambda_n P_n^{(\alpha+1,\beta)}(\cos\phi)$$

Therefore

$$S_n(1) - \mathrm{A} = 2^{(\alpha+\beta+1)}\lambda_n\int_0^{\pi}(\sin\tfrac{\phi}{2})^{2\alpha+1}(\cos\tfrac{\phi}{2})^{2\beta+1}[f(\cos\phi)-\mathrm{A}]\,\mathrm{P}_n^{\alpha+1,\beta}(\cos\phi)d\phi$$

$$= 2^{(\alpha+\beta+1)}\lambda_n\int_0^{\pi}F(\phi)P_n^{(\alpha+1,\beta)}(\cos\phi)dQ \tag{5.2}$$

The (N, q_n, r_n) means of the series (1.2) of the point $x = +1$ given by

$$t_n = \frac{1}{(q*r)_n}\sum_{k=0}^{n} q_k r_{n-k} s_{n-k}(1)$$

$$t_n - A = \frac{1}{(q*r)_n}\sum_{k=0}^{n} q_k r_{n-k}\{s_{n-k}(1) - A\}$$

$$= \frac{1}{(q*r)_n}\sum_{k=0}^{n} q_k r_{n-k}\,2^{(\alpha+\beta+1)}\lambda_{n-k}\int_0^{\pi}F(\phi)P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)d\phi$$

$$= \int_0^{\pi} f(\phi)N_n(\phi)d\phi \tag{5.3}$$

Where

$$N_n(\phi) = \frac{1}{(q*r)_n}(2)^{(\alpha+\beta+1)}\sum_{k=0}^{n} q_k r_{n-k}\lambda_{n-k}P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)$$

To prove the theorem we have to show that

$$I = \int_0^{\pi} F(\phi) N_n(\phi) d\phi = O(1), \text{ as } n \to \infty$$

$$I = \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-\frac{1}{n}} \int_{\pi-\frac{1}{n}}^{\pi} \right) F(\phi) N_n(\phi) d\phi$$

$$= I_1 + I_2 + I_3 + I_4 \text{ (say)} \qquad (5.4)$$

Now

$$|I_1| = \left| \int_0^{1/n} f(\phi) N_n(\phi) d\phi \right|$$

Using (4.1), we have

$$I_1 = O(n^{2\alpha+2}) O\left(\frac{n^{-2\alpha-2}}{\log n} \right)$$

$$= O\left(\frac{1}{\log n} \right)$$

$$= O(1) \text{ as } n \to \infty \qquad (5.5)$$

Next

$$|I_2| = \left| \int_{1/n}^{\delta} F(\phi) N_n(\phi) \right|$$

Using (4.2) we have

$$I_2 = \mathrm{O}\left(\int_{1/n}^{\delta} | f(\phi) | \frac{n^{(2\alpha+1)/2} (q*r)_{(1/\phi)}}{(q*r)_n} \left(\sin \frac{\phi}{2} \right)^{-(2\alpha+3)/2} d\phi \right)$$

$$+ \mathrm{O}\left(\int_{1/n}^{\delta} | f(\phi) | \, \mathrm{n}^{(2\alpha-1)/7} (\sin \tfrac{\phi}{2})^{-(2\alpha-5)/2} d\phi \right)$$

$$= I_{2,1} + I_{2,2} \qquad (5.6)$$

For given any $c > 0$, let be chosen so that

$$| f_1(\phi) | \le \frac{c\phi^{2\alpha+2}}{\log(1/\phi)}, \text{ for } 0 \le \phi \le \delta$$

Then

$$| I_{2,1} | \le \frac{m_n^{(2\alpha+1)/2}}{(\mathrm{q*r})_n} \int_{1/n}^{\delta} | f(\phi) | \frac{(q*r)_{(1/\phi)}}{\phi^{(2\alpha+3)/2}} d\phi$$

Where, we suppose M is used throughout the paper to denotes a positive constant, which may be different at each occurrence.

$$| I_{2,1} | = \frac{M_n^{(2\alpha+1)/2}}{(q*r)_n} \left\{ \left[\frac{F_1(\phi)(q*r)_{(1/\phi)}}{\phi^{(2\alpha+3)/2}} \right]_{1/n}^{\delta} - \int_{1/n}^{\delta} f_1(\phi) d \left[\frac{(q*r)(1/\phi)}{\phi^{(2\alpha+3)/2}} \right] \right\}$$

$$= I_{2,1,1} + I_{2,1,2} \qquad (5.7)$$

Now if $m(\delta)$ denotes a constant depending on δ , we have for fixed δ .

$$I_{2,1,1} = m(\delta)\frac{n^{(2\alpha+1)/2}}{(q*r)_n} + O\left(\frac{1}{\log n}\right)$$

$$= O(1) \text{ as } n \to \infty \tag{5.8}$$

And

$$|I_{2,1,2}| \le m_{\in}\frac{n^{(2\alpha+1)/2}}{(q*r)_n}\left[\int_{1/n}^{\delta}\frac{\phi^{2\alpha+2}}{\log\left(\frac{1}{\phi}\right)}d\left\{\frac{(q*r)_{(\frac{1}{\phi})}}{\phi^{(2\alpha+3)/2}}\right\}\right]$$

$$\le m_{\in}\frac{n^{(2\alpha+1)/2}}{(q*r)_n}\left[\int_{1/\delta}^{n}\frac{x^{-2\alpha-2}}{\log x}d\left\{(q*r)_{(x)}x^{(2\alpha+3)/2}\right\}\right]$$

$$\le m_{\in}\frac{n^{(2\alpha+1)/2}}{(q*r)_n}\int_{1/\delta}^{n}\frac{x^{-2\alpha-2}}{\log x}\left\{x^{(2\alpha+3)/2}d(q*r)_{(x)} + (2\alpha+3)x^{(2\alpha+1)/2}(q*r)_{(x)}dx\right\}$$

$$\le m_{\in}\frac{n^{(2\alpha+1)/2}}{(q*r)_n}\left[\int_{1/\delta}^{n}\frac{x^{(-2\alpha-1)/2}}{\log x}d(q*r)_{(n)} + (2\alpha+3)/2\int_{1/\delta}^{n}\frac{x^{(-2a-3)/2}}{\log x}[q*r]_{(x)}dx\right]$$

$$= m_{\in}\frac{n^{(2\alpha+1)/2}}{(q*r)_n}\left[J + ((2\alpha+3)/2)\mathrm{k}\right] \text{ (say)}$$

Since $(q*r)_{(x)}$ has a jump of $(q*r)_k$ at $x = k$ therefore

$$J = \sum_{k=a}^{n}\frac{(q*r)_k}{k^{(2\alpha+1)/2}\log k}$$

Where a is a fixed positive integer

But, since $(q*r)_k$ is non-negative, non increasing, $(k+1)(q*r)_k \le O(q*r)_k$ so

$$J = O\left(\sum_{k=a}^{n}\frac{(q*r)_k}{k^{(2\alpha+1)/2}\log k}\right)$$

Also

$$k \le \sum_{k=a-1}^{n-1}(q*r)_k\int_{k}^{k+1}\frac{x^{-(2\alpha+3)/2}}{\log x}dx$$

$$= O\left\{\sum_{k=a-1}^{n-1}\frac{(q*r)_k}{k^{(2\alpha+3)/2}\log k}\right\}$$

By (3.1) we have

$$|I_{2,1,1}| \le m_{\in} \tag{5.9}$$

Again

$$|I_{2,2}| \le m_n^{(2\alpha-1)/2}\int_{1/n}^{\delta}|f(\phi)|\phi^{(-2a-5)/2}d\phi$$

$$= n^{(2\alpha-1)/2}\{m[F_1(\phi)\phi^{(-2\alpha-5)/2}]_{1/n}^{\delta} + m\int_{1/n}^{\delta}F_1(\phi)\phi^{(-2a-7)/2}d\phi\}$$

$$= I_{2,2,1} + I_{2,2,2} \tag{5.10}$$

Now

$$I_{2,2,1} = m(\delta)n^{(2\alpha-1)/2} + O\left(\frac{1}{\log n}\right)$$

$$= O(1) \text{ as } n \to \infty \tag{5.11}$$

Also

$$|I_{2,2,2}| \le m_{\in} n^{(2\alpha-1)/2} \int_{1/n}^{\delta} \frac{\phi^{(2\alpha-3)/2}}{\log(1/\phi)} d\phi$$

$$\le m_{\in} n^{(2\alpha-1)/2} \int_{1/n}^{\delta} \frac{x^{(-2\alpha-1)/2}}{\log x} dx$$

$$= m_{\in} \text{ because } \alpha < \frac{1}{2} \tag{5.12}$$

Hence

$$I_2 = O(1) \text{ as } n \to \infty \tag{5.13}$$

Next

$$I_3 = O\left\{\frac{n^{(2\alpha+1)/2}}{(q*r)_n} \int_{\delta}^{\pi-(1/n)} \cos\left(\frac{\phi}{2}\right)^{(-2\beta-1)/2} |F(\phi) d\phi\right\} + O\left\{n^{(2\alpha-1)/2} \int_{\delta}^{\pi-(1/n)} (\cos\tfrac{1}{2}\phi)^{(-2\beta-3)/2} |F(\phi) d\phi\right\}$$

$$= I_{3,1} + I_{3,2} \quad \text{(say)} \tag{5.14}$$

Since $\alpha \ge \frac{-1}{2}$, we have

$$-\beta - \frac{1}{2} \ge -\beta - \alpha - 1$$

So that (4.8) Implies that

$$\int_{\delta}^{\pi} (\cos\tfrac{\phi}{2})^{(-2\beta-1)/2} |F(\phi)| d\phi < \infty$$

Hence $I_{3,1} = O\left(\frac{n^{(2\alpha+1)/2}}{(q*r)_n}\right)$

$$= O(1) \text{ as } n \to \infty \tag{5.15}$$

Now it is follows from (4.7) that, given any $\in > 0$, we can choosen $\eta > 0$, so that

$$\int_{\pi-\eta}^{\pi} (\cos(\phi/2))^{-\alpha-\beta-1} |F(\phi)| d\phi < \in \tag{5.16}$$

Thus, supposing that $n > \frac{1}{\eta}$, we have

$$n^{(2\alpha-1)/2} \int_{\pi-\eta}^{\pi-\frac{1}{n}} (\cos\tfrac{\phi}{2})^{(-2\beta-3)/2} |F(\phi)| d\phi$$

$$\leq n^{(2\alpha-1)/2}\{\cos\tfrac{1}{2}[\pi-(\tfrac{1}{n})]\}^{(2\alpha-1)/2}\int_{\pi-\eta}^{\pi-\frac{1}{n}}(\cos\tfrac{1}{2}\phi)^{-\alpha-\beta-1}|F(\phi)|\,d\phi$$

$$\leq 2\in$$

by (5.16), provided that n is sufficiently large. But, once η has been fixed.

$$n^{(2\alpha-1)/2}\int_{\delta}^{\pi-\eta}(\cos\tfrac{\phi}{2})^{-\alpha-\beta-1}|F(\phi)|\,d\phi$$

Is just a constant, and hence can be made $<\in$, by choosing n sufficiently large. Hence

$$I_3 = O(1) \tag{5.17}$$

Finally, since $\alpha+\beta+1>O$

$$I_4 = O\left(n^{\alpha+\beta+1}\int_{\pi-\frac{1}{n}}^{\pi}|F(\phi)|\,d\phi\right)$$

$$= O\left\{\int_{\pi-\frac{1}{n}}^{\pi}(\cos\tfrac{1}{2}\phi)^{-\alpha-\beta-1}|F(\phi)|\,d\phi\right\}$$

$$= O(1) \tag{5.18}$$

Using (5.5), (5.13), (5.17) and (5.18) we have

$$I = O(1)$$

This completes the proof of the theorem.

6. CONCLUSIONS

This theorem has more general result rather than the result of B.N. Pandey [7] that will enrich the literature on Jacobi summability theory.

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