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# On Theoretical Foundation of *p*-Version of General Ray Method for Solution of the Dirichlet Boundary Value Problems for Poisson Equation in Plane Domains

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**Abstract:** *The p-version of the General Ray (GR) method for approximate solution of the Dirichlet boundary value problem for the Partial Differential Equation (PDE) of Poisson is considered. GR-method consists in application of the Radon transform directly to the PDE and in reduction PDE to assemblage of Ordinary Differential Equations (ODE). GR-method presents the approximate solution of the Dirichlet boundary value problem for this type of equation by explicit analytical formulas that use the direct and inverse Radon transform. For the implementation of direct Radon transform to partial differential equations in limited domains of the plane it is necessary to have some properties related to the Radon transform, applied to the Laplace operator in limited domains. In this work we present such relations, and on the base of specific approximation of desired solution we give theoretical foundation GR-method as numerical method for considering problem.*

**Keywords:** *Partial Differential Equations, Boundary Value Problems, Radon Transform.*

## 1. Introduction

There are two main approaches for solving boundary value problems for partial differential equations in analytical form: the Fourier decomposition and the Green function method [1]. The Fourier decomposition is used, as the rule, only in theoretical investigations. The Green function method is the explicit one, but it is difficult to construct the Green function for the complex geometry of the considered domain $\Omega$ . The known numerical algorithms are based on the Finite Differences method, Finite Elements (Finite Volume) method and the Boundary Integral Equation method. Numerical approaches lead to solving systems of linear algebraic equations [2] that require a lot of computer time and memory.

A new approach for the solution of boundary value problems on the base of the General Ray Principle (*GRP*) was proposed by the author in [3], [4] for the stationary fields. *GRP* leads to explicit analytical formulas (*GR*-method) and fast algorithms, developed and illustrated by numerical experiments in [4] - [7] for solution of the direct and coefficient inverse problems for the equations of mathematical physics.

Here we consider the *p*-version of *GR*-method [6] based on application of the direct Radon transform [8] to the PDE [9] and present its theoretical justification for the plain domain.

## 2. General Ray Principle and *p*-Version of *GR*-Method for Plane Domain

The General Ray Principle (*GRP*) was proposed in [3], [4]. It includes construction for considering PDE an analogue as family of ODE describing the distribution of the desired function $u\ x,y$ along of “General Rays", which are presented by a straight line $l$ with the traditional Radon parameterization due a parameter $t$:

$$x = p\cos\varphi - t\sin\varphi\,,\ y = p\sin\varphi + t\cos\varphi\,.$$

Here $|p|$ is a length of the perpendicular from the center of coordinates to the line $l$, $\varphi \in \; 0,\pi$ is the angle between the axis *x* and this perpendicular.

Let us consider plane convex domain $\Omega$ with contour $\Gamma$, presented on the plane by formula $r = r_0(\alpha) > 0$ in polar coordinates, and consider the problem

$$\Delta u \;\; x,y \;\; = \psi(x,y), \quad (x,y) \in \Omega; \tag{1}$$

$$u \;\; x,y \;\; = f(x,y), \quad (x,y) \in \Gamma. \tag{2}$$

Here $r_0(\alpha)$, $\psi(x,y)$, $f(x,y)$ are known functions.

The *p*-version of the *GR*-method for 2D domain can be explained as the consequence of the next steps [6]:

1) reduce the problem to equivalent one with no homogeneous equation and homogeneous boundary condition:

$$\Delta u_0 \;\; x,y \;\; = \psi_0(x,y), \quad (x,y) \in \Omega; \tag{3}$$

$$u_0 \;\; x,y \;\; = 0, \quad (x,y) \in \Gamma; \tag{4}$$

2) extend the desired solution $u_0 \;\; x,y$ of the reduced problem and function $\psi_0(x,y)$ to all plane $R^2$, such that

$$u_0 \;\; x,y \;\; = 0, \psi_0(x,y) = 0, (x,y) \notin \Omega,$$

and describe its along on a general ray (a straight line $l$) by application of the Direct Radon Transform (DRT) operator [8] *R*:

$$\hat{u}_0 \;\; p,\varphi \;\; = R[u_0(x,y)],\; \hat{\psi}_0(p,\varphi) = R\,[\psi_0(x,y)];$$

3) construct the family of ODE on the variable *p* with respect the function $\hat{u}_0 \;\; p,\varphi$, supposing the fulfillment of formula (2) at the pp. 3 in [8]:

$$R[\Delta u_0(x,y)] = \frac{d^2\hat{u}_0 \;\; p,\varphi}{dp^2} \tag{5}$$

and applying the direct Radon transform to equation (3);

4) resolve of the constructed ODE with zero boundary conditions;
5) calculate $u_0 \;\; x,y$ as the inverse Radon transform of the obtained solution;
6) regress to the initial boundary condition, calculating the function $u \;\; x,y$ .

## 3. Application of Direct Radon Transform to Laplace Operator in Limited Domains on the Plane

For considering case the Radon transform of the function $u_0 \;\; x,y$ can be calculated in the next form:

$$\hat{u}_0 \;\; p,\varphi \;\; = \int_{-g_1 \;\; p}^{g_2 \;\; p} u_0 \;\; p\cos\varphi - \tau sen\varphi, psen\varphi + \tau\cos\varphi \;\; d\tau,$$

where

$$-r_0(\pi-\varphi) \le p \le r_0(\varphi), \quad 0 \le \varphi < \pi,$$
$$g_1(p,\varphi) = \sqrt{r_0^{\,2}(\alpha_1) - p^2},$$
$$g_2(p,\varphi) = \sqrt{r_0^{\,2}(\alpha_2) - p^2},$$

angles $\alpha_1$, $\alpha_2$ correspond to polar coordinates of points of intersection of the straight line ***l*** and $\Gamma$**.** Using parameterized function

$$\tilde{u}_0(p,\tau,\varphi) = u_0(p\cos\varphi - \tau sen\,\varphi, p\,sen\,\varphi + \tau\cos\varphi),$$

which satisfies the Leibniz derivative rule [10], by deductions similar ones presented in [11], we obtained formulas

$$R\left[\Delta u_0(x,y)\right](p,\varphi) = \frac{\partial^2 \hat{u}_0}{\partial p^2} -$$
$$- g_2{}'(p)\frac{\partial \tilde{u}_0(p, g_2(p),\varphi)}{\partial p} - g_1{}'(p)\frac{\partial \tilde{u}_0(p,-g_1(p),\varphi)}{\partial p}$$
$$- g_2{}''(p)\tilde{u}_0(p,g_2(p),\varphi) - g_1{}''(p)\tilde{u}_0(p,-g_1(p),\varphi)$$
$$-2\,sen\,\varphi\cos\varphi\left[\frac{\partial \tilde{u}_0(p, g_2(p),\varphi)}{\partial p} + \frac{\partial \tilde{u}_0(p,-g_1(p),\varphi)}{\partial p}\right]$$
$$-(\cos\varphi\, g_2{}'(p) + sen\,\varphi)\widetilde{\frac{\partial u_0}{\partial x}}(p, g_2(p),\varphi)$$
$$-(\cos\varphi\, g_1{}'(p) + sen\,\varphi)\widetilde{\frac{\partial u_0}{\partial x}}(p, -g_1(p),\varphi)$$
$$-(sen\,\varphi\; g_2{}'(p) + \cos\varphi)\widetilde{\frac{\partial u_0}{\partial y}}(p, g_2(p),\varphi)$$
$$-(sen\,\varphi\; g_1{}'(p) + \cos\varphi)\widetilde{\frac{\partial u_0}{\partial y}}(p, -g_1(p),\varphi),$$

where

$$\frac{\partial \tilde{u}_0(p, g_2(p),\varphi)}{\partial p} = \left(\cos\varphi\frac{\partial u_0(x,y)}{\partial x} + sen\,\varphi\frac{\partial u_0(x,y)}{\partial y}\right)\Bigg|_{\substack{x = p\cos\varphi - g_2(p)sen\,\varphi \\ y = p\,sen\,\varphi + g_2(p)\cos\varphi}};$$

$$\frac{\partial \tilde{u}_0(p, -g_1(p),\varphi)}{\partial p} = \left(\cos\varphi\frac{\partial u_0(x,y)}{\partial x} + sen\,\varphi\frac{\partial u_0(x,y)}{\partial y}\right)\Bigg|_{\substack{x = p\cos\varphi + g_1(p)sen\,\varphi \\ y = p\,sen\,\varphi - g_1(p)\cos\varphi}}.$$

From this result we can conclude that formula (5) is true for $u_0(x,y)$, if the extended function $u_0(x,y)$ has continues derivatives of the second order in all the plane and function and its first partial derivatives on *x* and *y* are equal to zero at the boundary curve. But frequently it can occur that corresponding derivatives are not continues, or are not equal to zero at the boundary curve. So, we need some auxiliary approximation of solution, $u_0(x,y)$, to which we can apply steps 3)-5) of the *p*-version of the *GR*-method.

## 4. APPROXIMATION OF SOLUTION AND FINAL FORMULA FOR ITS CALCULATION

To construct this approximation we will define subdomain $\Omega^{\varepsilon}$ as a part of $\Omega$, such that for every $\alpha$: $r < r_0(\alpha) - \varepsilon$. Let us consider the Hermit polynomial $P_5(\mathrm{x},\mathrm{y}) = \bar{P}_5(\mathrm{r},\alpha)$ of the order 5 in dependence of variable $r$ for every fixed $\alpha$, that satisfies to conditions

$$\bar{P}_5^{(k)}(r_0(\alpha)-\varepsilon,\alpha) = \frac{\partial^k u_0}{\partial r^k}(r_0(\alpha)-\varepsilon,\alpha),\ k = 0,1,2,$$

$$\bar{P}_5^{(k)}(r_0(\alpha)-\varepsilon/2,\alpha) = 0,\ k = 0,1,2.$$

The approximation is the next one:

$$u_o^{\varepsilon}(x,y) = \begin{cases} u_0(x,y),(x,y)\in\Omega^{\varepsilon}; \\ P_5(x,y),(x,y)\in\Omega^{\varepsilon/2}/\Omega^{\varepsilon}; \\ 0,(x,y)\in R^2/\Omega^{\varepsilon/2}; \end{cases} \tag{6}$$

Presented approximation $u_0^{\varepsilon}(x,y)$ has continues derivatives to the second order at the full plane, and out of the domain this function and its first and second derivatives are equal to zero. Let us define functions $\psi_0^{\varepsilon}(x,y) = \Delta u_0^{\varepsilon}(x,y)$, $(x,y)\in R^2$, $\hat{\psi}_0^{\varepsilon}(p,\varphi) = R[\psi_0^{\varepsilon}(x,y)]$.

**Lemma.** The next convergence is fulfilled:

$$\left\|\hat{\psi}_0^{\varepsilon}(p,\varphi)-\hat{\psi}_0(p,\varphi)\right\|_{L_2[\Omega]} \to 0,\ \varepsilon\to 0.$$

**Proof.** From formula (6) it follows that function $u_0^{\varepsilon}(x,y)$ has limited second derivatives at the domain and it is different from $u_0(x,y)$ only in sub domain $\Omega/\Omega^{\varepsilon}$. So, $\Delta u_0^{\varepsilon}(x,y)$ also is different from $\Delta u_0(x,y)$ only in sub domain $\Omega/\Omega^{\varepsilon}$. On the base of these properties and limitation of the norm of the Radon transform operator [2], we can write estimates:

$$\left\|\hat{\psi}_0^{\varepsilon}(p,\varphi)-\hat{\psi}_0(p,\varphi)\right\|_{L_2[\hat{\Omega}]} \le \|R\|_{L_2[R^2]\to L_2[R^2]}\left\|\psi_0^{\varepsilon}(x,y)-\psi_0(x,y)\right\|_{L_2[\Omega/\Omega^h]} \le C\varepsilon^{1/2},$$

where $C$ is some constant no dependent of $\varepsilon$. So, we finished proof of desired convergence.

**Supposition 1.** Let for some domain $\Omega_1$ the next property is fulfilled:

if function $g(x,y) = 0,(x,y)\notin\Omega_1$, so $\hat{g}(p,\varphi) = R[\mathrm{g}(\mathrm{x},\mathrm{y})](p,\varphi) = 0,(p,\varphi)\notin\hat{\Omega}_1$,

where $\hat{\Omega}_1$ is presentation of the considering domain in Radon coordinates.

**Theorem.** If Supposition 1 is fulfilled, then solution $u_0(x,y)$ of the problem (3) – (4) can be calculate by formulas:

$$u_0(x,y) = R^{-1}[\hat{\psi}_2(p,\varphi) - \frac{(p+r_o(\varphi-\pi))}{r_o(\varphi)+r_o(\varphi-\pi)}\hat{\psi}_2(r_o(\varphi),\varphi)] \tag{7}$$

$$\hat{\psi}_2(p,\varphi) = \int_{-r_o(\varphi-\pi)}^{p}\int_{-r_o(\varphi-\pi)}^{p}\hat{\psi}_0(p,\varphi)dp \tag{8}$$

**Proof.** Due to previous constructions, next equation is satisfied:

$$\Delta u_0^{\varepsilon}(x,y) = \psi_0^{\varepsilon}(x,y),\quad (x,y)\in R^2. \tag{9}$$

For function $u_0^\varepsilon(x,y)$ and its Radon transform $\hat{u}_0^\varepsilon(p,\varphi) = R[u_0^\varepsilon(x,y)]$, the relation (5) is fulfilled:

$$R[\Delta u_0^\varepsilon(x,y)] = \frac{d^2\hat{u}_0^\varepsilon(p,\varphi)}{dp^2}. \tag{10}$$

Hence, all steps of the *GR*-method scheme can be realized for $u_0^\varepsilon(x,y)$. Applying the Radon transform to equation (9), we have relations

$$\frac{d^2\hat{u}_0^\varepsilon(p,\varphi)}{dp^2} = \hat{\psi}_0^\varepsilon(p,\varphi),\ -\infty < p < \infty, 0 \le \varphi \le \pi, \tag{11}$$

where $\hat{u}_0^\varepsilon(p,\varphi)$ is equal to zero in the boundary points due to relation (6) and Supposition 1:

$$\hat{u}_0^\varepsilon(-r_o(\varphi-\pi),\varphi) = \hat{u}_0^\varepsilon(r_o(\varphi),\varphi) = 0,\ \ \varphi \in [o,\pi]. \tag{12}$$

So, $\hat{u}_0^\varepsilon(p,\varphi)$ can be presented as unique solution of the problem (11) – (12) by formulas:

$$\hat{u}_0^\varepsilon(p,\varphi) = \hat{\psi}_2^\varepsilon(p,\varphi) - \frac{(p + r_o(\varphi-\pi))}{r_o(\varphi) + r_o(\varphi-\pi)}\hat{\psi}_2^\varepsilon(r_o(\varphi),\varphi),$$

$$\hat{\psi}_2^\varepsilon(p,\varphi) = \int_{-r_o(\varphi-\pi)}^{p} \int_{-r_o(\varphi-\pi)}^{p} \hat{\psi}_0^\varepsilon(p,\varphi)dp.$$

Hence, for $u_0^\varepsilon(x,y)$ we justified the formula

$$u_0^\varepsilon(\mathrm{x},\mathrm{y}) = R^{-1}\left[\hat{\psi}_2^\varepsilon(p,\varphi) - \frac{(p + r_o(\varphi-\pi))}{r_o(\varphi) + r_o(\varphi-\pi)}\hat{\psi}_2^\varepsilon(r_o(\varphi),\varphi)\right]. \tag{13}$$

From **Lemma** it follows that function $\hat{\psi}_2^\varepsilon(p,\varphi)$ and its first derivative on variable *p* converge uniformly in domain $\Omega$ to function $\hat{\psi}_2(p,\varphi)$ and its derivative correspondently. The inverse Radon transform operator is limited from the space $C^1(\hat{\Omega})$ to the space $C(\Omega)$ [2]. These two reasons confirm convergence of the right hand side of the relation (13) to the right hand side of the relation (7). At the same time $u_0^\varepsilon(x,y)$ converges to $u_0(x,y)$ due to its construction with formula (6) and double smoothness of $u_0(x,y)$. This finishes our proof.

The Supposition 1 is obviously fulfilled for the domain as a circle with the center in the origin of coordinates.

## 5. Conclusion

Theoretical justification of the *p*-version of *GR*-method for the approximate solution of the Dirichlet boundary value problem for the Poisson equation in a plain domain is presented. It is important to mention that *p*-version of *GR*-method is realized as fast algorithms and program package in MATLAB system, and its realizability is illustrated by numerical experiments for different type of plane domains [6], [12].

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