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Stability Analysis of Three Species Model in Series Mutualism with Bionomic and Optimal Harvesting of Two Terminal Species

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Abstract: *This paper deals with a three species Syn-eco-system consisting of three species (S_1), (S_2) and (S_3) are in series the ecological interaction: Mutualism in pairs while the terminal species S_1 and S_3 are harvested. The possibility of existence of bio economic equilibrium is being discussed and an optimal harvesting policy is given using Pontryagin's maximum principle. Further some numerical examples are computed using Matlab.*

Keywords: *Series Mutualism, Optimal harvesting, Bionomic harvesting, Terminal species*

1. INTRODUCTION

There is an extensive study on several kinds of prey- predator interactions after it was initiated by Lotka [1] and Volterra [2]. Bionomics of natural resources has played a significant role in all these interactions. There is a strong impact of harvesting on the dynamic evolution of a population. In fishery, forestry, agriculture and wild life management, the exploitation of biological resources and harvesting of population species can be seen. The problems of predator-prey systems in the presence of harvesting were discussed by many authors and attention on economic policies from harvesting have also been analyzed. A detailed discussion on the issues and techniques associated with the bionomic exploitation of natural resources was given by Clark [3, 4]. A study on a class of predator-prey models under constant rate of harvesting of both species simultaneously was made by Brauer and Soudack [5, 6]. Multi-species harvesting models are also studied in detail by Chaudhuri [7, 8]. Models on the combined harvesting of a two species prey predator fishery have been discussed by Ragozin and Brown [9], Chaudhuri and Saha Ray [10]. K. Shiva Reddy et.al [12] and B. Ravindra Reddy [13, 14, 15] proposed the mathematical models for two and three species ecosystem with bionomic and optimal harvesting. They also investigated the stability concepts using various mathematical techniques. In this connection, a three species mathematical model in series mutualism based on the system of non-linear equations has been constructed. Biological and Bionomical equilibria of the system are derived.

2. MATHEMATICAL MODEL

The model equations in this problem are as follows

(i) Equation for the growth rate of (S_1):

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 + a_{12}N_1N_2 - q_1E_1N_1 \tag{2.1}$$

(ii) Equation for the growth rate of (S_2):

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_2N_1 + a_{23}N_2N_3 \tag{2.2}$$

(iii) Equation for the growth rate of (S_3):

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{32}N_3N_2 - q_3E_3N_3 \tag{2.3}$$

Notation Adopted:

$N_i(t)$: Population density of the species S_i at time t, i=1, 2, 3.

a_i : Natural growth rate of S_i, i = 1,2,3

a_{ii} : Decrease rate of S_i due to its own insufficient resources i = 1, 2, 3.

a_{12} : Increase rate of the first species (S_1) due to inhibition by the second species (S_2),

a_{21} : Increase rate of the second species (S_2) due to (S_1)

a_{23} : Increase rate of the second species (S_2) due to (S_3),

a_{32} : Increase rate of the third species (S_3) due to (S_2)

$K_i = a_i/\alpha_{ii}$: Carrying capacities of S_i, i = 1, 2, 3.

q_i : Catch ability coefficient of the species (S_i), i=1, 3

E_i : Effort applied to harvest the first species (S_i), i=1, 3

$q_1E_1N_1$, $q_3E_3N_3$ are the catch-rate functions based on the catch-per-unit-effort hypothesis.

The variables N_1, N_2 and N_3 are non-negative and the model parameters $a_i, K_i, a_{ii}, a_{12}, a_{21}$, a_{23}, a_{32} are assumed to be non-negative constants.

Further $a_1 > q_1E_1, a_3 > q_3E_3$

3. EQUILIBRIUM STATES

The system under investigation has eight equilibrium states defined by $\frac{dN_i}{dt} = 0,\ i = 1, 2, 3$ and these are given hereunder.

I. Fully washed out state:

E1: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0$

II. States in which only one species survives while the other two are washed out

E2: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3 - q_3E_3}{a_{33}}$; E3: $\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0$

E4: $\overline{N_1} = \frac{a_1 - q_1E_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0$

III. States in which two species survive and the third washed out

E5: $\overline{N_1} = 0,\ \overline{N_2} = \frac{a_2a_{33} + a_{23}(a_3 - q_3E_3)}{a_{22}a_{33} - a_{23}a_{32}},\ \overline{N_3} = \frac{a_2a_{32} + a_{22}(a_3 - q_3E_3)}{a_{22}a_{33} - a_{23}a_{32}}$

This state exists only when $a_{22}a_{33} - a_{23}a_{32} > 0$

E6: $\overline{N_1} = \frac{a_1 - q_1E_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3 - q_3E_3}{a_{33}}$

E7: $\overline{N_1} = \frac{a_2a_{12} + (a_1 - q_1E_1)a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \overline{N_2} = \frac{a_2a_{11} + (a_1 - q_1E_1)a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \overline{N_3} = 0$

This state exists only when $a_{11}a_{22} > a_{12}a_{21}$

IV. The co-existent state (or) Normal steady state

E8: $\overline{N_1} = \frac{(a_1 - q_1E_1)\ \ a_{22}a_{33} - a_{32}aa_{23}\ \ + a_{12}a_{23}(a_3 - q_3E_3) + a_2a_{12}a_{33}}{a_{11}a_{22}a_{33} + a_{12}a_{21}a_{23} - a_{11}a_{23}a_{32}}$,

$$\overline{N_2} = \frac{a_2 a_{11} a_{33} + a_{21} a_{33}(a_1 - q_1 E_1) + a_{11} a_{23}(a_3 - q_3 E_3)}{a_{11} a_{22} a_{33} + a_{12} a_{21} a_{23} - a_{11} a_{23} a_{32}} ,$$

$$\overline{N_3} = \frac{(a_3 - q_3 E_3)\ \ a_{22} a_{33} + a_{12} a_{12}\ \ + a_2 a_{11} a_{32} - a_{32} a_{21}(a_1 - q_1 E_1)}{a_{11} a_{22} a_{33} + a_{13} a_{31} a_{22} - a_{12} a_{21} a_{33}}$$

This state exists only when $(a_{33} + a_{13} a_{31} a_{22}) > a_{12} a_{21} a_{33}$.

4. BIO ECONOMIC ASPECT AT INTERIOR EQUILIBRIUM POINT

The concept of bionomic equilibrium is a union of those of biological equilibrium as well as economic equilibrium. Biological equilibrium is given by $\frac{dN_i}{dt} = 0$, i=1, 2, 3.

By definition, the bionomic equilibrium is said to be achieved when the selling price of the harvested biomass equals to the total cost price utilized in harvesting it.

Let c_i be the harvesting cost per unit effort for $S_{i,}\ i = 1,3$ and p_i be the price per unit biomass of $S_{i,}\ i = 1,3$. The net revenue or economic rent at any time instant is then given by $R = R_1 + R_3$, where $R_1 = (p_1 q_1 N_1 - c_1) E_1$, $R_3 = (p_3 q_3 N_3 - c_3) E_3$

Here R_i represent net revenue for $S_{i,}\ i = 1,3$. The bionomic equilibrium $(N_1)_\infty, (N_2)_\infty, (N_3)_\infty,\ E_1\ {}_\infty,\ E_3\ {}_\infty$ satisfies the following equations.

$$a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2 - q_1 E_1 N_1 = 0 \tag{4.1}$$

$$a_2 N_2 - a_{22} N_2^2 + a_{21} N_2 N_1 + a_{23} N_2 N_3 = 0 \tag{4.2}$$

$$a_3 N_3 - a_{33} N_3^2 + a_{32} N_2 N_3 - q_3 E_3 N_3 = 0 \tag{4.3}$$

The revenue returns (R) on first and third species taken together will be

$$R = \ p_1 q_1 N_1 - c_1\ \ E_1 + \ p_3 q_3 N_3 - c_3\ \ E_3 \tag{4.4}$$

The cases would arise while determining the bionomic equilibrium.

Case (i): if $c_3 > p_3 q_3 N_3$ then the cost is greater than revenue for third species then its harvesting would come to a halt (E_3=0). Only the harvesting of first species remains operational. $c_1 < p_1 q_1 N_1$

$$N_1\ {}_\infty = \frac{c_1}{p_1 q_1} \tag{4.5}$$

$$N_2\ {}_\infty = \frac{1}{a_{22} a_{33} - a_{23} a_{32}} \left(a_2 a_{33} + a_{21} a_{33} \frac{c_1}{p_1 q_1} + a_3 a_{23} \right) \tag{4.6}$$

$$N_3\ {}_\infty = \frac{1}{a_{22} a_{33} - a_{23} a_{32}} \left(a_3 a_{22} + a_2 a_{32} + a_{32} a_{21} \frac{c_1}{p_1 q_1} \right) \tag{4.7}$$

$$E_1\ {}_\infty = \frac{1}{q_1} \left\{ a_1 - a_{11} \frac{c_1}{p_1 q_1} + a_{12}\ \ N_2\ {}_\infty \right\} \tag{4.8}$$

The condition, for this $E_1\ {}_\infty > 0$ to be positive definite, is that

$$a_1 + a_{12}\ \ N_2\ {}_\infty\ > a_{11} \frac{c_1}{p_1 q_1} \tag{4.9}$$

Case (ii): If $c_1 > p_1 q_1 N_1$ then the cost is greater than revenue for first species then its harvesting would come to a halt (E_1=0). Only the harvesting of third species remains operational ($c_3 < p_3 q_3 N_3$)

$$N_{3\ \infty} = \frac{c_3}{p_3 q_3} \quad (4.10)$$

$$N_{1\ \infty} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}}\left(a_1 a_{22} + a_{21}\frac{c_3}{p_3 q_3} + a_2 a_{12}\right) \quad (4.11)$$

$$N_{2\ \infty} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}}\left(a_2 a_{11} + a_1 a_{21} - a_{11}\frac{c_3}{p_3 q_3}\right) \quad (4.12)$$

Now substituting $N_{1\ \infty}$, $N_{2\ \infty}$, $N_{3\ \infty}$ in equations (4), (5) and (6) we get

$$E_{3\ \infty} = \frac{1}{q_2}\left\{a_3 + a_{32}\ N_{2\ \infty} - a_{33}\frac{c_3}{p_3 q_3}\right\} \quad (4.13)$$

Now $E_{3\ \infty} > 0$, where the following condition

$$a_3 + a_{32}\ N_{2\ \infty} > a_{33}\frac{c_3}{p_3 q_3} \quad (4.14)$$

Case (iii): if $c_1 > p_1 q_1 N_1$, $c_3 > p_3 q_3 N_3$ then the cost is greater than the revenue for the both species and species-harvesting will come to a total closer.

Case (iv): if $c_1 < p_1 q_1 N_1$, $c_3 < p_3 q_3 N_3$ the cost is less than revenue return on the harvesting of the both the species, the system becomes operational to yield profit.

The bionomic equilibrium $(N_1)_\infty, (N_2)_\infty, (N_3)_\infty,\ E_{1\ \infty},\ E_{3\ \infty}$ is the positive solution of the system (4.1 – 4.3)

Solving these equations, we get

$$N_{1\ \infty} = \frac{c_1}{p_1 q_1},\ N_{3\ \infty} = \frac{c_3}{p_3 q_3},\ N_{2\ \infty} = \frac{1}{a_{22}}\left(a_2 + a_{21}\frac{c_1}{p_1 q_1} + a_{23}\frac{c_3}{p_3 q_3}\right) \quad (4.15)$$

$$E_{1\ \infty} = \frac{1}{q_1}\left\{a_1 - a_{11}\frac{c_1}{p_1 q_1} + a_{12}\ N_{2\ \infty}\right\} \quad (4.16)$$

$$E_{3\ \infty} = \frac{1}{q_3}\left\{a_3 + a_{32}\ N_{2\ \infty} - a_{33}\frac{c_3}{p_3 q_3}\right\} \quad (4.17)$$

$$E_{1\ \infty} > 0,\quad E_{3\ \infty} > 0 \quad (4.18)$$

$$a_1 + a_{12}\ N_{2\ \infty} > a_{11}\frac{c_1}{p_1 q_1},\quad a_3 + a_{32}\ N_{2\ \infty} > a_{33}\frac{c_3}{p_3 q_3} \quad (4.19)$$

Thus the bionomic equilibrium $(N_1)_\infty, (N_2)_\infty, (N_3)_\infty,\ E_{1\ \infty},\ E_{3\ \infty}$ exits if the conditions (4.18) and (4.19) hold.

5. Optimal Harvesting Policy

The present target is to select the harvesting policy that maximizes the present value J of continuous time stream of revenues given by

$$J = \int_0^\infty e^{-\delta t}\ (p_1 q_1 N_1 - c_1)E_1(t) + (p_3 q_3 N_3 - c_3)E_3(t)\ dt \quad (5.1)$$

where δ denotes the instantaneous annual rate of discount. Intentionally we have to maximize (5.1) subject to the state equations (2.1)–(2.3) by adopting Pontryagin's maximum principle.

The control variable $E_i(t)$ is subjected to the constrains $0 \le E_i(t) \le (E)_{\max}$.

The Hamiltonian for the problem is given by

$$H = e^{-\delta t}\ \ (p_1q_1N_1 - c_1)E_1\ +\ (p_3q_3N_3 - c_3)E_3\ + \lambda_1\left[a_1N_1 - a_{11}N_1^2 + a_{12}N_1N_2 - q_1E_1N_1\right]$$
$$+ \lambda_2\left[a_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 + a_{23}N_2N_3\right] + \lambda_3\left[a_3N_3 - a_{33}N_3^2 + a_{32}N_3N_2 - q_3E_3N_3\right] \tag{5.2}$$

where λ_1, λ_2 and λ_3 are the adjoint variables.

Consider the switching functions

$$\mu_1(t) = e^{-\delta t}\ \ (p_1q_1N_1 - c_1)E_1\ - \lambda_1q_1E_1N_1 . \tag{5.3}$$

and

$$\mu_2(t) = e^{-\delta t}\ \ (p_3q_3N_3 - c_3)E_2\ - \lambda_3q_3E_3N_3 \tag{5.4}$$

The optimal control will be a combination of extreme controls and the singular control.
The optimal control function $E_1(t)$ and $E_3(t)$ that maximizes H must satisfy the following conditions.

$$E_1 = (E_1)_{\max,}\quad \text{when } \mu_1(t) > 0 \text{ i.e } \lambda_1(t)e^{\delta t} < p_1 - \frac{c_1}{q_1\overline{\overline{N_1}}} \tag{5.5}$$

$$E_1 = 0\quad \text{when } \mu_1(t) < 0 \quad \text{i.e } \lambda_1(t)e^{\delta t} > p_1 - \frac{c_1}{q_1\overline{\overline{N_1}}} \tag{5.6}$$

and

$$E_3 = (E_3)_{\max,}\quad \text{when } \mu_2(t) > 0 \text{ i.e } \lambda_3(t)e^{\delta t} < p_3 - \frac{c_3}{q_3\overline{\overline{N_3}}} \tag{5.7}$$

$$E_3 = 0\quad \text{when } \mu_2(t) < 0 \quad \text{i.e } \lambda_3(t)e^{\delta t} > p_3 - \frac{c_3}{q_3\overline{\overline{N_3}}} \tag{5.8}$$

Thus the optimal harvesting policy is

$$E_1(t) = \begin{cases} (E_1)_{\max} & ;\mu_1(t) > 0 \\ 0 & ;\mu_1(t) < 0 \\ E^* & ;\mu_1(t) = 0 \end{cases} \tag{5.9}$$

and

$$E_3(t) = \begin{cases} (E_3)_{\max} & ;\mu_2(t) > 0 \\ 0 & ;\mu_2(t) < 0 \\ E^* & ;\mu_2(t) = 0 \end{cases} \tag{5.10}$$

By Pontryagin's maximum principle,

$$\frac{\partial H}{\partial E_1} = 0 ; \frac{\partial H}{\partial E_2} = 0\ ;\ \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1}\ ;\ \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2}\ \text{ and } \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial N_3} \tag{5.11}$$

$$\frac{\partial H}{\partial E_1} = 0 \Rightarrow e^{-\delta t}\ \ p_1q_1N_1 - c_1\ + \lambda_1\ -q_1N_1\ = 0 \Rightarrow \lambda_1 = e^{-\delta t}\left[p_1 - \frac{c_1}{q_1\overline{\overline{N_1}}}\right] \tag{5.12}$$

$$\frac{\partial H}{\partial E_2} = 0 \Rightarrow e^{-\delta t}\ \ p_3q_3N_3 - c_3\ + \lambda_3\ -q_3N_3\ = 0 \Rightarrow \lambda_3 = e^{-\delta t}\left[p_3 - \frac{c_3}{q_3\overline{\overline{N_3}}}\right] \tag{5.13}$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} = -\left[e^{-\delta t}p_1q_1E_1 + \lambda_1\ \ a_1 - 2a_{11}N_1 + a_{12}N_2 - q_1E_1\ + \lambda_2\ \ a_{21}N_2\ \right] \tag{5.14}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2} = \lambda_1a_{12}N_1 - \lambda_2\ \ a_2 + a_{21}N_1 - 2a_{22}N_2 + a_{23}N_3 \tag{5.15}$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial N_3} = -\left[e^{-\delta t}p_3q_3E_3 + \lambda_2\ \ a_{23}N_2\ + \lambda_3\ \ a_3 + a_{32}N_2 - 2a_{33}N_3 - q_3E_3\ \right] \tag{5.16}$$

After simplification we get

$$\frac{d\lambda_1}{dt} = -e^{-\delta t} p_1 q_1 E_1 + \lambda_1 a_{11} \overline{N_1} - \lambda_2 a_{21} \overline{N_2} \tag{5.17}$$

$$\frac{d\lambda_2}{dt} = -\lambda_1 a_{12} \overline{N_1} + \lambda_2 a_{22} \overline{N_2} + \lambda_3 a_{23} \overline{N_3} \tag{5.18}$$

$$\frac{d\lambda_3}{dt} = -e^{-\delta t} p_3 q_3 E_3 - \lambda_2 a_{23} \overline{N_2} + \lambda_3 a_{33} \overline{N_3} \tag{5.19}$$

From (5.12), (5.13) and (5.18)

$$\frac{d\lambda_2}{dt} - \lambda_2 a_{22} \overline{N_2} = -a_{12} \overline{N_1} \left[p_1 - \frac{c_1}{q_1 \overline{N_1}} \right] e^{-\delta t} + a_{23} \overline{N_3} \left[p_3 - \frac{c_3}{q_3 \overline{N_3}} \right] e^{-\delta t}$$

i.e, $\frac{d\lambda_2}{dt} - a_{22} \overline{N_2} \lambda_2 = A_1 \, e^{-\delta t}$ (5.20)

where $A_1 = -a_{12} \overline{N_1} \left[p_1 - \frac{c_1}{q_1 \overline{N_1}} \right] + a_{23} \overline{N_3} \left[p_3 - \frac{c_3}{q_3 \overline{N_3}} \right]$

The solution of which can be obtained as $\lambda_2 = \frac{-A_1}{\delta + a_{22} \overline{N_2}} e^{-\delta t}$ (5.21)

From (5.21) and (5.17),

$$\frac{d\lambda_1}{dt} = -e^{-\delta t} p_1 q_1 E_1 + \lambda_1 a_{11} \overline{N_1} - a_{22} \overline{N_2} \frac{-A_1}{\delta + a_{22} \overline{N_2}} e^{-\delta t}$$

i.e, $\frac{d\lambda_1}{dt} - \lambda_1 a_{11} \overline{N_1} = - A_2 \, e^{-\delta t}$ (5.22)

The solution of which can be obtained as $\lambda_1 = \frac{A_2}{\delta + a_{11} \overline{N_1}} e^{-\delta t}$ (5.23)

where $A_2 = \left(p_1 q_1 E_1 - a_{22} \overline{N_2} \frac{A_1}{\delta + a_{22} \overline{N_2}} \right)$

From (5.21) and (5.19),

$$\frac{d\lambda_3}{dt} - a_{33} \overline{N_3} \lambda_3 = -e^{-\delta t} p_3 q_3 E_3 - a_{23} \overline{N_2} \frac{-A_1}{\delta + a_{22} \overline{N_2}} e^{-\delta t}$$

i.e, $\frac{d\lambda_3}{dt} - a_{33} \overline{N_3} \lambda_3 = - A_3 \, e^{-\delta t}$ (5.24)

The solution of which can be obtained as $\lambda_3 = \frac{A_3}{\delta + a_{33} \overline{N_3}} e^{-\delta t}$ (5.25)

where $A_3 = \left(p_3 q_3 E_3 - a_{23} \overline{N_2} \frac{A_1}{\delta + a_{22} \overline{N_2}} \right)$

From (5.12) and (5.23), we get a singular path, $\frac{A_2}{\delta + a_{11} \overline{N_1}} e^{-\delta t} = e^{-\delta t} \left[p_1 - \frac{c_1}{q_1 \overline{N_1}} \right]$ from which we obtain $\frac{A_2}{\delta + a_{11} \overline{N_1}} = \left(p_1 - \frac{c_1}{q_1 \overline{N_1}} \right)$ (5.26)

From (5.13) and (5.25), we get a singular path,

$$\frac{A_3}{\delta + a_{33}\overline{N_3}} e^{-\delta t} = e^{-\delta t}\left[p_3 - \frac{c_3}{q_3\overline{N_3}} \right]$$

From which we obtain $\dfrac{A_3}{\delta + a_{33}\overline{N_3}} = \left(p_3 - \dfrac{c_3}{q_3\overline{N_3}} \right)$ (5.27)

Thus from (5.26) and (5.27), we write as,

$$F(\overline{N_1}) = \left(p_1 - \frac{c_1}{q_1\overline{N_1}} \right) - \frac{A_2}{\delta + a_{11}\overline{N_1}} = 0 \tag{5.28}$$

$$G(\overline{N_3}) = \left(p_3 - \frac{c_3}{q_3\overline{N_3}} \right) - \frac{A_3}{\delta + a_{33}\overline{N_3}} = 0 \tag{5.29}$$

There exists a unique positive root $\overline{N_1} = (N_1)_\delta$ of $F(\overline{N_1}) = 0$ in the interval $0 < N_{1\ \infty} < K_1$ if the following inequalities hold: $F(0) < 0, F(K_1) > 0$, $F^{'}(\overline{N_1}) > 0$ for $\overline{N_1} > 0$. Similarly there exists a unique positive root $\overline{N_3} = (N_3)_\delta$ if $G(\overline{N_3}) = 0$ in the interval $0 < N_{3\ \infty} < K_3$ If the following inequalities hold: $G(0) < 0, G(K_3) > 0$, $G^{'}(\overline{N_3}) > 0$ for $\overline{N_3} > 0$

For $\overline{N_1} = N_{1\ \infty}$, $\overline{N_3} = N_{3\ \infty}$, we get

$$N_{2\ \infty} = \frac{1}{a_{22}}\left(a_2 + a_{21}\frac{c_1}{p_1 q_1} + a_{23}\frac{c_3}{p_3 q_3} \right) \tag{5.30}$$

$$E_{1\ \infty} = \frac{1}{q_1}\left(a_1 - a_{11}\frac{c_1}{p_1 q_1} + a_{12}\ N_{2\ \infty} \right) \tag{5.31}$$

$$E_{3\ \infty} = \frac{1}{q_3}\left(a_3 + a_{32}\ N_{2\ \infty} - a_{33}\frac{c_3}{p_3 q_3} \right) \tag{5.32}$$

Hence once the optimal equilibrium $N_{1\ \delta}$, $N_{2\ \delta}$, $N_{3\ \delta}$ is determined, the optimal harvesting effort $E_{1\ \infty}$ and $E_{3\ \infty}$ can be determined. From (5.21), (5.23) and (5.25) we observe that $\lambda_i(t)e^{\delta t} (i = 1,2,3)$ is independent of time is an optimum equilibrium. Hence they satisfy the transversality condition at ∞. That is they remain bounded as $t \to \infty$.

From (5.26) and (5.27) we have

$$\frac{A_2}{\delta + a_{11}\overline{N_1}} = \left(p_1 - \frac{c_1}{q_1\overline{N_1}} \right) \to 0 \quad \text{as } t \to \infty \quad \text{and}$$

$$\frac{A_3}{\delta + a_{33}\overline{N_3}} = \left(p_3 - \frac{c_3}{q_3\overline{N_3}} \right) \to 0 \text{ as } t \to \infty$$

Thus the total economic revenue

$$(N_1)_\infty, (N_2)_\infty, (N_3)_\infty,\ E_{1\ \infty}, t \ = 0$$

$$(N_1)_\infty, (N_2)_\infty, (N_3)_\infty,\ E_{3\ \infty}, t \ = 0$$

This implies that an infinite discount rate leads to the total economic revenue tending to zero, and hence the system would remains closed.

6. Numerical Simulations

(1) Let a_1=3,α_{11}=0.5,α_{12}=0.5,q_1=0.35,E_1= 10,a_2=4,α_{21}=0.84,α_{22}=2.4, α_{23}=0.02,a_3=3.5, α_{32}=0.5, α_{33}=2**,** q_3 =0.3,E_3 =12, N_1 =15, N_2 =20 and N_3 =10

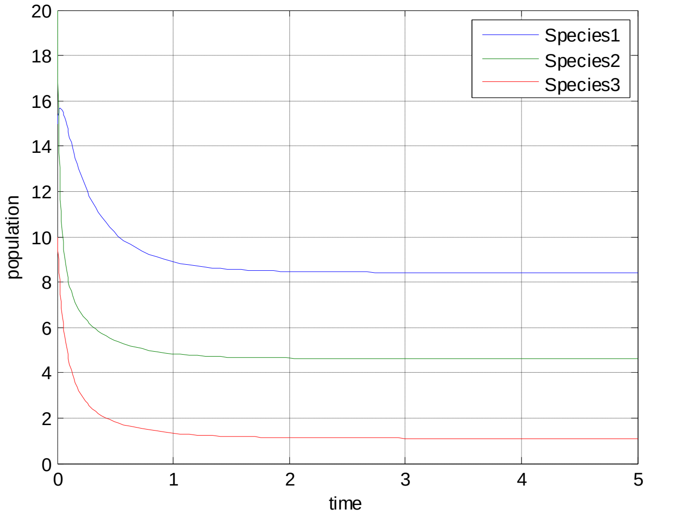

Fig 6.1. *Population growth rate Variations verses time.*

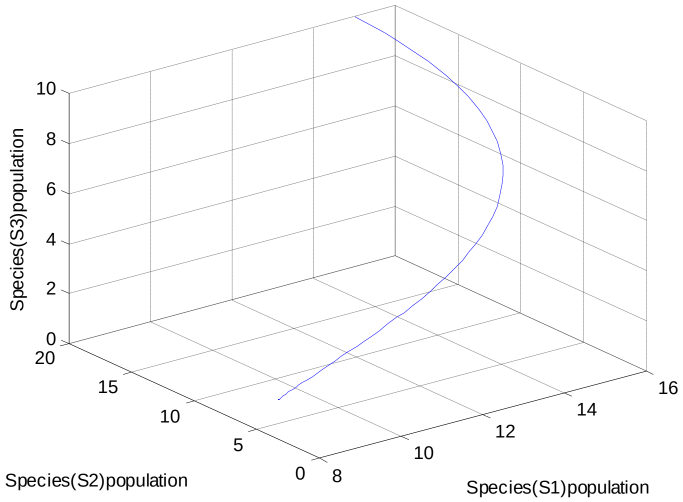

Fig 6.2. *Phase-space trajectories corresponding to the stabilities of the population*

(2) Let $a_1=2, \alpha_{11}=1, \alpha_{12}=0.35, q_1=0.01, E_1=8, a_2=2, \alpha_{21}=0.1, \alpha_{22}=0.6$, $\alpha_{23}=0.2, a_3=2.4$, $\alpha_{32}=0.4$, $\alpha_{33}=0.4$, $q_3=0.69, E_3=6, N_1=6$, $N_2=8$ and $N_3=10$

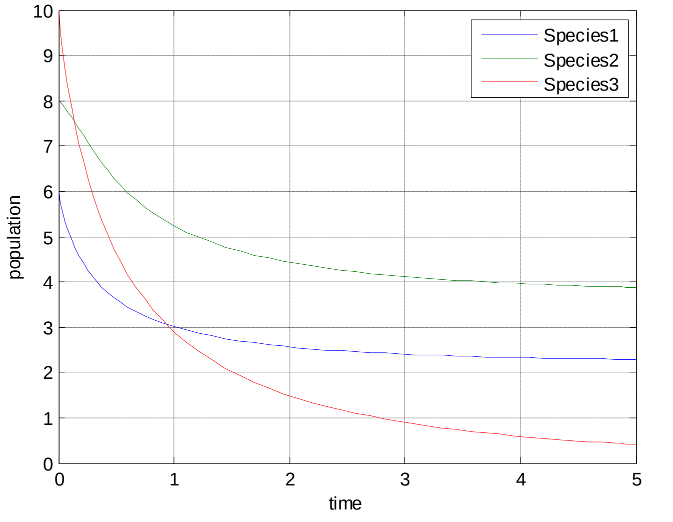

Fig 6.3. *Population growth rate Variations verses time.*

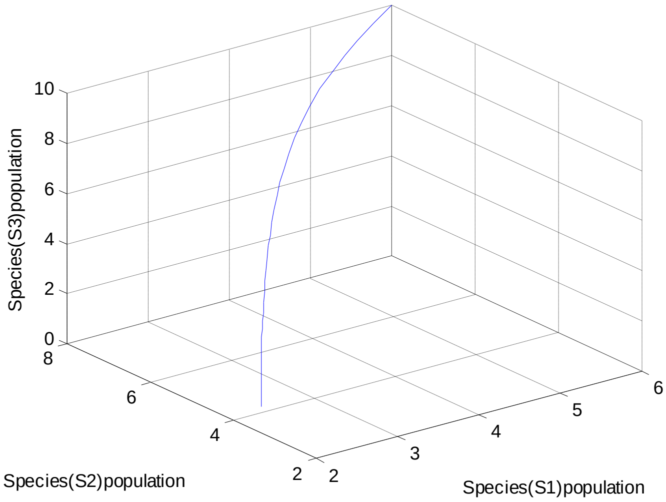

Fig 6.4. *Phase-space trajectories*

7. Concluding Remarks

The bionomic equilibrium has been analyzed followed by the determination of optimal harvesting policy by employing Pontryagin's Maximum Principle [11]. At the steady state, the harvesting cost per unit effort is equal to the marginal profit of the effort. It is found that even under continuous harvesting of the terminal species, the population may be maintained at an appropriate equilibrium level. Some numerical examples are also computed using Matlab.

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