International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 11, November 2014, PP 892-913
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Degree of Approximation of Conjugate Fourier Series of Fractional Mean of Function of Bounded Variation by (Z,δ,β) Method

Samira Behera

Department of Mathematics
Assam University, Silchar
Silchar, India
samirabehera1998@gmail.com

Abstract: *In this paper we study degree of Approximation of conjugate Fourier series of fractional mean of function of bounded variation by (Z,δ,β) method.*

Keywords: *Conjugate Fourier series, function of bounded variation*

Definition and Notation

Let f be periodic with period 2π and Lebesgue integrable on $[-\pi, \pi]$.

Let $$\frac{a_0}{2}+\sum_{1}^{\infty}(a_k\cos kx+b_k\sin kx) \tag{1.1}$$

be the Fourier series of f at t=x and let $s_n(f,x)$ represents the n_{th} partial sums of the series (1.1)

Let $$\sum_{1}^{\infty}(b_k\cos kx-a_k\sin kx) \tag{1.2}$$

be the conjugate Fourier series of f at t=x and let $\tilde{s}_n(x)$; represents the nth partial sums of the series (1.2).

Let $W_n(f, x)$ and $\tilde{W}_n(f,x)$ be the Norlund or (N,p) mean of $s_n(x)$ and $\tilde{s}_n(x)$; where $p_n\geq 0$ and

$P_n=p_0+p_1+.....+p_n\to\infty$ as $n\to\infty$

Then

$$W_n(f,x)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\,s_k(f,x) \tag{1.3}$$

$$\tilde{W}(f,x)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\tilde{s}_k(f,x) \tag{1.4}$$

Let $\tilde{\sigma}_n^{\delta}(\tilde{f},x)$ be Cesaro or (C, δ) mean of order δ of $\tilde{s}_n(f,x)$ for $\delta>-1$.

$$\tilde{\sigma}_n^{\delta}(\tilde{f},x)=\frac{1}{A_n^{\delta}}\sum_{k=0}^{n}A_{n-k}^{\delta-1}\;\tilde{s}_k(f,x) \tag{1.5}$$

where the binomial coefficient (A_n^{δ}) is defined by

$$\sum_{n=0}^{\infty} A_n^{\delta} x^n = \frac{1}{(1-x)^{\delta+1}} \qquad \left(|x|<1\right) \tag{1.6}$$

Das and Mohapatra [3] have studied a special (N,p) method which they called generalized harmonic-Cesaro method (Z, δ,β) and which is generated by the sequence $\{p_n\}: p_n = A_n^{\delta-1,\beta}$, determined by the identity.

$$(1-z)^{-\delta-1}(\log\frac{a}{1-z})^{\beta} = \sum_{n=0}^{\infty} A_n^{\delta,\beta} z^n \,(|z|<1)$$

Where a≥2, is a fixed constant.

Let $\tilde{K}_n^{\delta,\beta}(f,x)$ denotes the (Z, δ,β) transform of the sequence of partial sums $\{\tilde{s}_n(f,x)\}$ of the Fourier series. Then

$$\tilde{K}_n^{\delta,\beta}(f,x) = \frac{1}{A_n^{\delta,\beta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1,\beta} \tilde{s}_k(f,x);$$

We use the following additional notations

$$\psi(t) = \psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\} \tag{1.7}$$

$V_0^t(\psi)$ = the total variation of ψ on [0, t]

$$\tilde{D}_n(t) = \left(\cos {t}/{2} - \cos(n+\frac{1}{2})t\right) / 2\sin {t}/{2} \tag{1.8}$$

$$\psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \; \psi(u)du, \qquad \alpha > 0 \tag{1.9}$$

$$\psi_0(t) = \psi(t) \tag{1.10}$$

$$\psi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Psi_\alpha(t) \qquad , \alpha\geq 0 \tag{1.11}$$

$$h=[\alpha] \tag{1.12}$$

$$\tilde{K}_n^{\delta,\beta}(t) = \frac{1}{A_n^{\delta,\beta}} \sum_{\nu=0}^{n} A_n^{\delta-1,\beta} \tilde{D}_\nu(t) \tag{1.13}$$

$$\tilde{J}(n,u) = \frac{1}{\Gamma(1-\alpha)} \int_u^{\pi/n} (t-u)^{-\alpha} \tilde{K}_n^{\delta,\beta}(t)dt \tag{1.14}$$

$$\tilde{J}_1(n,u) = \frac{1}{\Gamma(1+h-\alpha)} \int_u^{\pi/n} (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^h \tilde{K}_n^{\delta,\beta}(t)dt \tag{1.15}$$

$$H_n^{\delta,\beta}(t) = \frac{1}{A_n^{\delta,\beta}} \sum_{\nu=0}^{n} A_n^{\delta-1,\beta} \frac{\cos(\nu+\frac{1}{2})t}{2\sin {t}/{2}} \tag{1.16}$$

$$\tilde{H}(n,u)=\frac{1}{\Gamma(1-\alpha)}\int_u^{\pi}(t-u)^{-\alpha}H_n^{\delta,\beta}(t)dt \tag{1.17}$$

$$\tilde{H}_1(n,u)=\frac{1}{\Gamma(1+h-\alpha)}\int_u^{\pi}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^h H_n^{\delta,\beta}(t)dt \tag{1.18}$$

$$\tilde{Q}(n,u)=\frac{1}{\Gamma(1-\alpha)}\int_u^{\pi}v^{\alpha}\frac{d}{dv}\tilde{H}(n,v)dv \tag{1.19}$$

$$\tilde{Q}_1(n,u)=\frac{1}{\Gamma(1-\alpha)}\int_u^{\pi}v^{\alpha}\frac{d}{dv}\tilde{H}_1(n,v)dv \tag{1.20}$$

$$\tilde{A}_n^{\delta}(f,x)=\tilde{\sigma}_n^{\delta}(f,x)-\tilde{f}\left(x,\pi/n\right) \tag{1.21}$$

$$\tilde{A}_n^{\delta,\beta}(f,x)=H_n^{\delta,\beta}(f,x)-\tilde{f}\left(x,\pi/n\right) \tag{1.22}$$

INTRODUCTION

It is well known that (See [5] Vol. I p.59)

If $f\in BV[-\pi,\pi]$, then

$$\lim_{n\to\infty}\left(\tilde{s}_n(f,x)-\tilde{f}\left(x,\pi/n\right)\right)=0 \tag{2.1}$$

Mazhar and AL-Budaiwi[4] obtained a sharper version of this result by showing that n≥1

$$\left|\left(\tilde{s}_n(f.x)-\tilde{f}\left(x,\pi/n\right)\right)\right|\le\frac{3.3}{n}\sum_{k=1}^{n}V_0^{\pi/k}(\psi) \tag{2.2}$$

The objective of the paper is to generalize and extend the above results and work with less stringent condition $\psi_\alpha(t)\in BV$ in place of $\psi(t)\in BV$ by using (Z,δ,β) method. It may be noted that similar work for Fourier series has been investigated by the present authors (See [7]) with $\phi_\alpha\in BV$, where ϕ_α is the Rieman Liouville integral mean of

$$\phi(t)=\{f(x+t)+f(x-t)-f(x+0)-f(x-0)\}$$

In short we prove the theorem as follows

Theorem. 1

Let $0\le\alpha<1$, $\beta\in R$. If $\psi_\alpha(t)\in BV[0,\pi]$ and $\delta+1>\alpha$ then

$$\tilde{A}_n^{\delta,\beta}(f,x)=O(1)\left\{\frac{1}{(\log n)^{\beta}n^{\delta+1-\alpha}}\sum_{k=1}^{n}k^{\delta-\alpha}\log^{\beta}(ak/2)V_0^{\pi/k}(\psi_\alpha)\right\}$$
$$+O(1)\left\{\frac{1}{n^{1-\alpha}}\sum_{k=1}^{n}k^{-\alpha}V_0^{\pi/k}(\psi_\alpha)\right\}$$

Theorem 2

Let $\alpha\ge1$. If $\psi_\alpha(t)\in BV[0,\pi]$ and $\delta>\alpha$ then

$$\tilde{A}_n^{\delta,\beta}\left(f,x\right)=\frac{O(1)}{(\log n)^{\beta}n^{\delta+1-\delta}}\sum_{k=1}^{n}k^{\delta-\alpha}\log^{\beta}kV_0^{\pi/k}(\psi_\alpha)$$

$$+\frac{O(1)}{(\log n)^{\beta}n^{\delta-\alpha}}\sum_{k=1}^{n}k^{\delta-\alpha-1}\log^{\beta}kV_0^{\pi/k}(\psi_\alpha)$$

To prove the Theorems we need the following Lemmas

Lemma 1

If $\delta>-1$ and F(u) is slowly varying, then

$$\sum_{v=1}^{n}v^{\delta}F(v)\simeq\frac{n^{\delta+1}}{\delta+1}\ F(n)$$

Lemma 2.

Let E(n,r,t)= $\dfrac{1}{A_n^{\delta,\beta}}\sum_{v=0}^{n}A_{n-v}^{\delta-1,\beta}r^r e^{ivt}$, r≥0 (3.1)

Let r=0,1,2,……………., 0<t<π and n>t^{-1}π. Then

for $-1<\delta<1$

E(n,r,t)= $O(t^{-\delta}(\log\frac{d}{t})^{\beta}n^{r-\delta}(\log n)^{-\beta})$ (3.2)

for $\delta\geq 1$

E(n,r,t)= $O(t^{-r-1}n^{-1})+O(t^{-\delta}(\log\frac{d}{t})^{\beta}n^{r-\delta}(\log n)^{-\beta})$ (3.3)

Where d is a suitable constant greater than π.

Note: Chandra and Dikshit have stated case (i) for $0<\delta<1$ but the result still holds for $-1<\delta<1$ as their line of argument also valid for the range $-1\leq\delta\leq 0$.

Lemma 3

For $a\geq 2,\beta\in R,-1<\delta<1$, we write

$$R=R(t)=\left[\log^2\left(\frac{a}{2\sin t/2}\right)+\left(\frac{\pi-t}{2}\right)^2\right]^{1/2}$$

$$\phi=\phi(t)=\tan^{-1}\left(\frac{t-\pi}{2\log\left(\frac{a}{2\sin t/2}\right)}\right)$$

Then for $0\leq t\leq\pi$

i) $H_n^{\delta,\beta}(t)=\dfrac{R^{\beta}\cos\left[\left(n+\frac{1}{2}+\frac{\delta}{2}\right)t-\frac{\pi\delta}{2}+\phi\beta\right]}{n^{\delta}(\log n)^{\beta}(2\sin t/2)^{\delta+1}}+\dfrac{O(1)}{n(2\sin t/2)^2},\delta\neq-1,-2...\ ,\beta\in R$ (3.4)

ii) $H_n^{\delta,\beta}(t)=O(1)\left\{\dfrac{(\log\frac{a\pi}{2t})^{\beta}}{n^{\delta+1}(\log n)^{\beta}t^{\delta+1}}+\dfrac{1}{nt},\quad \delta>-1,\beta\in R\right.$ (3.5)

part (i) is contained in lemma 3 of [6].

As $A_n^{\delta,\beta}\simeq\dfrac{n^{\delta}(\log n)^{\beta}}{\Gamma(\delta+1)},\ \delta\neq-1,\ \beta\in R$

And $R^{\beta}(t)=O(1)(\log\dfrac{a\pi}{2t})^{\beta}$ part (ii) follows from part(i)

Lemma 4

Let r=0,1,2,……………., $\beta\in R,$ 0<t<π ,then

$$(\frac{d}{dt})^r\tilde{K}_n^{\delta,\beta}(t)\ =O(n^{r+1}) \tag{3.6}$$

For $-1<\delta<1$ and n>t^{-1}π.

$$(\frac{d}{dt})^r H_n^{\delta,\beta}(t)\ =O\left\{t^{-1-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta}\right\} \tag{3.7}$$

For $\delta\geq1$ and n>t^{-1}π.

$$(\frac{d}{dt})^r H_n^{\delta,\beta}(t)\ =O\left(\frac{1}{nt^{r+2}}\right)+O\left\{t^{-1-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta}\right\} \tag{3.8}$$

For $\delta\geq1$, $\delta\leq r+1$ and n>t^{-1}π.

$$(\frac{d}{dt})^r H_n^{\delta,\beta}(t)\ =O\left\{t^{-1-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta}\right\} \tag{3.9}$$

Proof.

Part (i) is trivial.

We write

C(n,t)= $\dfrac{1}{A_n^{\delta,\beta}}\displaystyle\sum_{k=0}^{n}A_{n-k}^{\delta-1,\beta}coskt$

S(n,t)= $\dfrac{1}{A_n^{\delta,\beta}}\displaystyle\sum_{k=0}^{n}A_{n-k}^{\delta-1,\beta}\sin kt$

$$C^{(r)}(n,t)=\left(\frac{d}{dt}\right)^r C(n,t)$$

$$S^{(r)}(n,t)=\left(\frac{d}{dt}\right)^r S(n,t)$$

By lemma 3, we have for **r=0,1,2,…,** $\beta\in R,$ $0<t\leq\pi$ and n>t^{-1}π.

$$\left.\begin{matrix} C^{(r)}(n,t) \\ S^{(r)}(n,t) \end{matrix}\right\} = O\left\{ t^{-\delta} \log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta} \right\}, \quad -1<\delta<1 \tag{3.10}$$

(ii) $$\left.\begin{matrix} C^{(r)}(n,t) \\ S^{(r)}(n,t) \end{matrix}\right\} = O(t^{-r-1}n^{-1}) + O\left\{ t^{-\delta} \log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta} \right\}, \ \delta \geq 1 \tag{3.11}$$

Now

$$(\frac{d}{dt})^r H_n^{\delta,\beta}(t) = (\frac{d}{dt})^r \frac{1}{A_n^{\delta,\beta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1,\beta} \left\{ \frac{1}{2}\cot\frac{1}{2}t\cos kt - \frac{1}{2}\sin kt \right\}$$

$$= (\frac{d}{dt})^r \left[\left(\frac{1}{2}\cot\frac{1}{2}t \right) C(n,t) - \frac{1}{2} S(n,t) \right] \tag{3.12}$$

Writing w(t) = $\frac{1}{2}\cot\frac{1}{2}t$, $w^{(r)}(t) = \left(\frac{d}{dt}\right)^r w(t)$ and applying Leibnitz theorem, we get

$$(\frac{d}{dt})^r H_n^{\delta,\beta}(t) = \sum_{r=0}^{p} \binom{r}{p} C^{(p)}(n,t) w^{(r-p)}(t) + \frac{1}{2} S^{(r)}(n,t) \tag{3.13}$$

Here we shall consider the cases $\delta \geq 1$ and $-1<\delta<1$ separately.

We first proceed to proved part (iii).

Using (3.12) for $C^{(r)}(n,t)$ and the fact that $w^{(s)}(t) = O(t^{-s-1})$,we get

$$\sum_{r=0}^{p} \binom{r}{p} S^{(p)}(n,t) w^{(r-p)}(t)$$

$$= \sum_{r=0}^{p} \left[O\left(t^{-p-1}n^{-1}\right) + O\left\{ t^{-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta} \right\} \right] O\left(t^{p-r-1}\right)$$

$$= \sum_{r=0}^{p} O\left(t^{-r-2}n^{-1}\right) + \sum_{r=0}^{p} O\left\{ t^{-\delta-1}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta}(nt)^{p-r} \right\}$$

$$= O\left(t^{-r-2}n^{-1}\right) + O\left(t^{-\delta-1}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta} \right) \tag{3.14}$$

Since $(nt)^{p-r} \leq \frac{1}{\pi^{r-p}}$ whenever $n>t^{-1}\pi$ and $p \leq r$.

Collecting the results from (3.13) and (3.14) and applying (3.11) for the estimate of $C^{(r)}(n,t)$, we get

$$(\frac{d}{dt})^r H_n^{\delta,\beta}(t) = O\left(\frac{1}{nt^{r+2}}\right) + O\left\{ t^{-1-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta} \right\}$$

And this completes the proof of part (iii).

We omit the proof of part (ii) as it can be proved by adopting the above lines of arguments and applying (3.10) instead of (3.11) for the estimates of $S^{(r)}(n,t)$ and $C^{(r)}(n,t)$ at appropriate stages.

For part (iv),we have

$$t^{-r-2}n^{-1}=t^{-\delta-1}n^{r-\delta}(nt)^{\delta-r-1}$$

$$\le t^{-1-\delta}n^{r-\delta}\pi^{\delta-r-1}\quad \text{(since } t\ge\frac{\pi}{n}\text{ and } \delta\le \text{r+1)}$$

And hence estimates given in part (iii) takes the following form:

$$H_n^{\delta,\beta}(t)=O(t^{-1-\delta}n^{r-\delta})+O\left\{t^{-1-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta}\right\}$$

$$=O\left\{t^{-1-\delta}\log^{\beta}(\frac{d}{t})(n^{r-\delta})(\log n)^{-\beta}\right\}$$

Lemma 5.

For $u\ge\frac{\pi}{n},0\le\alpha<1,\delta+1>\alpha$

$$H(n,u)=O(1)\frac{\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}u^{\delta+1}(\log n)^{\beta}}+O(1)\frac{1}{n^{1-\alpha}u}$$

For $u\ge\frac{\pi}{n},\alpha\ge1,\delta>\alpha$

$$\tilde{H}_1(n,u)=O(1)\frac{\log^{\beta}\left(d/u\right)}{n^{\delta+1-\alpha}u^{\delta+1}(\log n)^{\beta}}+O(1)\frac{\log^{\beta}\left(d/u\right)}{n^{\delta-\alpha}u(\log n)^{\beta}}$$

For $0\le\alpha<1,\delta+1>\alpha$

$$\tilde{Q}(n,u)=O(1)\frac{\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}u^{\delta+1-\alpha}(\log n)^{\beta}}+O(1)\frac{1}{(nu)^{1-\alpha}}$$

For $\alpha\ge1,\delta>\alpha$

$$\tilde{Q}_l(n,u)=O(1)\frac{\log^{\beta}\left(d/u\right)}{n^{\delta+1-\alpha}u^{\delta+1-\alpha}(\log n)^{\beta}}+O(1)\frac{\log^{\beta}\left(d/u\right)}{n^{\delta-\alpha}u^{\delta-\alpha}(\log n)^{\beta}}$$

Proof.

See lemma 3(i)

$$H_n^{\delta,\beta}(t)=\frac{R^{\beta}\cos\left[\left(n+\frac{1}{2}+\frac{\delta}{2}\right)t-\frac{\pi\delta}{2}+\phi\beta\right]}{n^{\delta}(\log n)^{\beta}(2\sin{t/2})^{\delta+1}}+\frac{O(1)}{n(2\sin{t/2})^{2}},\delta\ne-1,-2\ldots,\qquad(3.15)$$

it follows that

$$\Gamma(1\text{-}\alpha).\tilde{H}(n,u)=\int_u^{\pi}(t-u)^{-\alpha}H_n^{\delta,\beta}(t)\,dt=\left(\int_u^{u+\frac{1}{n}}+\int_{u+\frac{1}{n}}^{\pi}\right)(t-u)^{-\alpha}H_n^{\delta,\beta}(t)\,dt$$

$$=J_1+J_2\qquad\text{(say)}\qquad(3.16)$$

where

$$J_1 = \frac{O(1)\log^{\beta}\left(a\pi/2u\right)}{n^{\delta}u^{\delta+1}(\log n)^{\beta}}\int_u^{u+\frac{1}{n}}\left(t-u\right)^{-\alpha}dt + \frac{O(1)}{nu^2}\int_u^{u+\frac{1}{n}}\left(t-u\right)^{-\alpha}dt$$

$$= \frac{O(1)\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}u^{\delta+1}(\log n)^{\beta}} + \frac{O(1)}{n^{2-\alpha}u^2} \qquad (3.17)$$

By lemma 3(i) and application of mean value theorem, we get

$$J_2 = \frac{1}{A_n^{\delta,\beta}}\int_{u+\frac{1}{n}}^{n}(t-u)^{-\alpha}\frac{R^{\beta}(t)\cos\left\{\left(n+\frac{1}{2}+\frac{\delta}{2}\right)t-\frac{\pi\delta}{2}+\Phi\beta\right\}}{\left(2\sin t/2\right)^{\delta-1}}dt$$

$$+\frac{O(1)}{n}\int_{u+\frac{1}{n}}^{\pi}\left(t-u\right)^{-\alpha}\frac{dt}{\left(2\sin t/2\right)^{2}}$$

$$= \frac{1}{A_n^{\delta,\beta}}\left(\frac{1}{n}\right)^{-\alpha}\frac{R^{\beta}\left(u+\frac{1}{n}\right)}{\left\{2\sin\frac{1}{2}\left(u+\frac{1}{n}\right)\right\}^{\delta+1}}\int_{u+\frac{1}{n}}^{n}\cos\left\{\left(n+\frac{1}{2}+\frac{\delta}{2}\right)t-\frac{\pi\delta}{2}+\Phi\beta\right\}dt$$

$$+O(1)\frac{1}{n}\left(\frac{1}{n}\right)^{-\alpha}\int_{u+\frac{1}{n}}^{u''}\frac{dt}{t^2} \qquad \left(\mathrm{u}+\frac{1}{\mathrm{n}}<u',u''\le\pi\right)$$

$$= O(1)\frac{\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}u^{\delta+1}(\log n)^{\beta}} + O(1)\frac{1}{n^{1-\alpha}u} \qquad (3.18)$$

Collecting the above results (3.17),(3.18)and (3.16) and using the fact that $u > \frac{\pi}{n}$, we have

$$\tilde{H}\left(n,u\right) = O(1)\frac{\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}u^{\delta+1}(\log n)^{\beta}} + O(1)\frac{1}{n^{1-\alpha}u} \qquad \left(\mathrm{u}\ge\pi/\mathrm{n}\right)$$

Hence the Lemma 5(i).

Proof of Lemma 5(ii)

We have

$$\Gamma\left(1+\mathrm{h}\text{-}\alpha\right)\tilde{H}_1\left(n,u\right) = \int_u^{\pi}\left(t-u\right)^{h-\alpha}\left(\frac{d}{dt}\right)^{h}H_n^{\delta,\beta}\left(t\right)dt$$

$$= \left(\int_u^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{\pi}\right)\left(t-u\right)^{h-\alpha}\left(\frac{d}{dt}\right)^{h}H_n^{\delta,\beta}\left(t\right)dt$$

$= K_1 + K_2$ (say) (3.19)

Now

$$K_1 = \int_u^{u+\frac{1}{n}}\left(t-u\right)^{h-\alpha}\left(\frac{d}{dt}\right)^{h}H_n^{\delta,\beta}\left(t\right)dt$$

by lemma 4(iv)

$$= \frac{O(1)}{n^{\delta-h}(\log n)^{\beta}} \int_{u}^{u+\frac{1}{n}} \frac{\left(t-u\right)^{h-\alpha} \log^{\beta}\left(d/t\right)}{t^{\delta+1}} dt$$

$$= \frac{O(1)}{n^{\delta-h}u^{\delta+1}(\log n)^{\beta}} \frac{\log^{\beta}\left(d/u\right)}{n^{1+h-\alpha}}$$

$$= \frac{O(1)\log^{\beta}\left(d/u\right)}{n^{\delta+1-\alpha}u^{\delta+1}(\log n)^{\beta}} \quad , \tag{3.20}$$

and

$$K_2 = \int_{u+\frac{1}{n}}^{\pi} \left(t-u\right)^{h-\alpha} \left(\frac{d}{dt}\right)^{h} H_n^{\delta,\beta}\left(t\right) dt$$ by lemma 4(iv)

$$= \frac{O(1)}{n^{\delta-h}(\log n)^{\beta}} \int_{u+\frac{1}{n}}^{\pi} \frac{\left(t-u\right)^{h-\alpha} \log^{\beta}\left(d/t\right)}{t^{\delta+1}} dt$$

$$= \frac{O(1)}{n^{\delta-h}(\log n)^{\beta}} \frac{1}{n^{h-\alpha}} \int_{u+\frac{1}{n}}^{\xi'} \frac{\log^{\beta}\left(d/t\right)dt}{t^{\delta+1}} dt \qquad (\mathrm{u}+\frac{1}{\mathrm{n}} < \xi' < \pi)$$

$$= \frac{O(1)\log^{\beta}\left(d/u\right)}{n^{\delta-\alpha}u^{\delta}(\log n)^{\beta}} \tag{3.21}$$

Collecting (3.20),(3.21),and (3.19) we have the result.

Proof of Lemma 5(iii)

By integrating by parts

$$\tilde{Q}\left(n,u\right) = \int_{u}^{\pi} v^{\alpha} \frac{d}{dv} \tilde{H}\left(n,v\right) dv$$

$$= \left[v^{\alpha}\tilde{H}\left(n,v\right)\right]_{v=u}^{\pi} - \alpha\int_{u}^{\pi} v^{\alpha-1}\tilde{H}\left(n,v\right) dv$$

$$= -u^{\alpha}\tilde{H}\left(n,v\right) - \alpha\int_{u}^{\pi} v^{\alpha-1}\tilde{H}\left(n,v\right) dv$$

$$= -u^{\alpha}\tilde{H}(n,v) + \frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}} \int_{u}^{\pi} \frac{v^{\alpha-1}\log^{\beta}\left(a\pi/2v\right)}{v^{\delta+1}} dv$$

$$+ \frac{O(1)}{n^{1-\alpha}} \int_{u}^{\pi} \frac{v^{\alpha-1}}{v} dv$$

$$= \frac{O(1)\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}(\log n)^{\beta}\ u^{\delta+1-\alpha}} + \frac{O(1)}{n^{1-\alpha}u^{1-\alpha}}$$

$$+ \frac{O(1)\log^{\beta}\left(a\pi/2u\right)}{n^{\delta+1-\alpha}(\log n)^{\beta}u^{\delta+1-\alpha}} + \frac{O(1)}{n^{1-\alpha}u^{1-\alpha}}$$

$$= \frac{O(1)\log^{\beta}\left(a\pi/2u\right)}{\left(nu\right)^{\delta+1-\alpha}(\log n)^{\beta}} + \frac{O(1)}{\left(nu\right)^{1-\alpha}}$$

Hence the proof of Lemma 5(iii)

Proof of Lemma 5(iv)

By integrating by parts

$$\tilde{Q}_1(n,u) = \int_u^{\pi} v^{\alpha} \frac{d}{dv} \tilde{H}_1(n,v) dv$$

$$= \left[v^{\alpha} \tilde{H}_1(n,v) \right]_{v=u}^{\pi} - \alpha \int_u^{\pi} v^{\alpha-1} \tilde{H}_1(n,v) dv$$

$$= -u^{\alpha} \tilde{H}_1(n,u) - \alpha \int_u^{\pi} v^{\alpha-1} \tilde{H}_1(n,v) dv$$

$$= \frac{O(1)\log^{\beta}(d/u)}{n^{\delta+1-\alpha} u^{\delta+1-\alpha} (\log n)^{\beta}} + \frac{O(1)\log^{\beta}(d/u)}{n^{\delta-\alpha} u^{\delta-\alpha} (\log n)^{\beta}} + \frac{O(1)}{n^{\delta+1-\alpha} (\log n)^{\beta}} \int_u^{\pi} \frac{v^{\alpha-1}}{v^{\delta-1}} \log^{\beta}(d/v) dv$$

$$+ \frac{O(1)}{(\log n)^{\beta} n^{\delta-\alpha}} \int_u^{\pi} \frac{v^{\alpha-1}}{v^{\delta}} \log^{\beta}(d/v) dv$$

$$= \frac{O(1)\log^{\beta}(d/u)}{(\log n)^{\beta} (nu)^{\delta+1-\alpha}} + \frac{O(1)\log^{\beta}(d/u)}{(\log n)^{\beta} (nu)^{\delta-\alpha}}$$

using Lemma 5(ii) Lemma 5(iv) follows.

Lemma 6.

For $0 \le \alpha < 1$, $\delta + 1 > \alpha$

(i) $$V_0^{\pi/n}(\psi_{\alpha}) = \frac{O(1)}{n^{\delta+1-\alpha} \log^{\beta} n} \sum_{k=1}^{n} k^{\delta-\alpha} \log^{\beta}(\frac{ak}{2}) V_0^{\pi/k}(\psi_{\alpha})$$

(ii) $$V_0^{\pi}(\psi_{\alpha}) = O(1) \sum_{k=1}^{n} k^{\delta-\alpha} \log^{\beta}(ak/2) V_0^{\pi/k}(\psi_{\alpha})$$

(iii) $$\frac{1}{n^{\delta+1-\alpha} (\log n)^{\beta}} \int_{\pi/n}^{\pi} \log^{\beta}(a\pi/2u) \frac{|d\psi_{\alpha}(u)|}{u^{\delta+1-\alpha}} du = \frac{O(1)}{n^{\delta+1-\alpha} (\log n)^{\beta}} \sum_{k=1}^{n} k^{\delta-\alpha} \log^{\beta}(ak/2) V_o^{\pi/k}(\Psi_{\alpha})$$

(iv) $$\frac{1}{n^{1-\alpha}} \int_{\pi/n}^{\pi} \frac{|d\psi_{\alpha}(u)|}{u^{1-\alpha}} = \frac{O(1)}{n^{1-\alpha}} \sum_{k=1}^{n} k^{-\alpha} V_0^{\pi/k}(\psi_{\alpha})$$

Proof.

Lemma 6(i) follows from

$$\sum_{k=1}^{n} \log^{\beta}(ak/2) k^{\delta-\alpha} V_0^{\pi/k}(\psi_{\alpha}) \ge V_0^{\pi/n}(\psi_{\alpha}) \sum_{k=1}^{n} \log^{\beta}(ak/2) k^{\delta-\alpha} \cong V_0^{\pi/n}(\psi_{\alpha}) \frac{n^{\delta+1-\alpha}}{\delta+1-\alpha}$$

Proof of Lemma 6(ii) is trivial

Proof of Lemma 6(iii)

Integrating by parts

$$\int_{\pi/n}^{\pi}\log^{\beta}\left(a\pi/2u\right)\frac{|d\psi_{\alpha}(u)|}{u^{\delta+1-\alpha}}$$

$$\leq\int_{\pi/n}^{\pi}\log^{\beta}\left(a\pi/2u\right)\frac{|d\psi_{\alpha}\left(u\right)|}{u^{\delta+1-\alpha}}\leq\int_{\pi/n}^{\pi}\log^{\beta}\left(a\pi/2u\right)\frac{dV_{0}^{u}\left(\psi_{\alpha}\right)}{u^{\delta+1-\alpha}}$$

$$=\log^{\beta}\left(\frac{a}{2}\right)\frac{V_{0}^{\pi}\left(\psi_{\alpha}\right)}{\pi^{\delta+1-\alpha}}-\frac{V_{0}^{\pi/n}\left(\psi_{\alpha}\right)\log^{\beta}\left(an/2\right)}{\left(\frac{\pi}{n}\right)^{\delta+1-\alpha}}$$

$$+\int_{\pi/n}^{\pi}\frac{\left(\beta\log^{\beta-1}\left(\frac{a\pi}{2u}\right)u^{-1}u^{\delta+1-\alpha}-(\delta+1-\alpha)u^{\delta-\alpha}\log^{\beta}\left(\frac{a\pi}{2u}\right)V_{0}^{u}(\psi_{\alpha})\right)du}{u^{2(\delta+1-\alpha)}}$$

$$=\log^{\beta}\left(\frac{a}{2}\right)\frac{V_{0}^{\pi}\left(\psi_{\alpha}\right)}{\pi^{\delta+1-\alpha}}-\frac{\log^{\beta}\left(an/2\right)V_{0}^{\pi}\left(\psi_{\alpha}\right)n^{\delta+1-\alpha}}{\pi^{\delta+1-\alpha}}$$

$$+O(1)\int_{\pi/n}^{\pi}\log^{\beta}\left(a\pi/2u\right)V_{0}^{\pi}\left(\psi_{\alpha}\right)\frac{V_{0}^{u}\left(\psi_{\alpha}\right)\log^{\beta}\left(an/2\right)}{\left(u\right)^{\delta+2-\alpha}} \tag{3.22}$$

Putting u=π/t in the last integral we get

$$\int_{\pi/n}^{\pi}\frac{\log^{\beta}\left(a\pi/2u\right)V_{0}^{u}\left(\psi_{\alpha}\right)}{u^{\delta+2-\alpha}}du=\frac{1}{\pi^{\delta+1-\alpha}}\int_{1}^{n}t^{\delta-\alpha}\log^{\beta}\left(at/2\right)V_{0}^{\pi/t}\left(\psi_{\alpha}\right)dt$$

$$\leq\frac{1}{\pi^{\delta+1-\alpha}}\sum_{k=1}^{n-1}V_{0}^{\pi/k}\left(\psi_{\alpha}\right)\int_{k}^{k+1}\log^{\beta}\left(at/2\right)t^{\delta-\alpha}dt$$

$$=O(1)\sum_{k=1}^{n}k^{\delta-\alpha}V_{0}^{\pi/k}\left(\psi_{\alpha}\right)$$

Hence

$$\frac{1}{(\log n)^{\beta}n^{\delta+1-\alpha}}\ \cdot\alpha\int_{\pi/n}^{\pi}\frac{\log^{\beta}\left(a\pi/2u\right)|d\psi_{\alpha}\left(u\right)|}{u^{\delta+1-\alpha}}$$

$$=\frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n}k^{\delta-\alpha}\log^{\beta}\left(ak/2\right)V_{0}^{\pi/k}\left(\psi_{\alpha}\right)$$

Similarly Lemma 6(iv) follows

Lemma 7.

For α≥1, δ>α

(i) $$V_{0}^{\pi/n}\left(\psi_{\alpha}\right)=\frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n}k^{\delta-\alpha}\log^{\beta}kV_{0}^{\pi/k}\left(\psi_{\alpha}\right)$$

(ii) $V_0^{\pi}\left(\psi_\alpha\right)=O(1)\sum_{k=1}^{n}k^{\delta-\alpha}\log^{\beta}kV_0^{\pi/k}\left(\psi_\alpha\right)$

(iii) $\frac{1}{n^{\delta+1-\alpha}(\log n)^{\beta}}\int_{\pi/n}^{\pi}\frac{\log^{\beta}\left(d/u\right)|\,d\psi_\alpha\left(u\right)|}{u^{\delta+1-\alpha}}=\frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n}k^{\delta-\alpha}\log^{\beta}kV_0^{\pi/k}\left(\psi_\alpha\right)$

(iv) $\frac{1}{(\log n)^{\beta}n^{\delta-\alpha}}\int_{\pi/n}^{\pi}\frac{\log^{\beta}\left(d/u\right)|\,d\psi_\alpha\left(u\right)|}{u^{\delta-\alpha}}=\frac{O(1)}{n^{\delta-\alpha}}\sum_{k=1}^{n}k^{\delta-\alpha-1}\log^{\beta}kV_0^{\pi/k}\left(\psi_\alpha\right)$

Proof.

Similar to Lemma 6, Lemma 7 could be verified.

Lemma 8 with h= [α],

(i) $\left[\sum_{k=1}^{h}(-1)^k\,\Psi_k\left(t\right)\left(\frac{d}{dt}\right)^{k-1}\tilde{K}_n^{\delta,\beta}\left(t\right)\right]_{t=0}^{t=\pi/n}=O(1)V_0^{\pi/n}\left(\psi_\alpha\right)$

(ii) $\left[\sum_{k=1}^{h}(-1)^k\,\Psi_k\left(t\right)\left(\frac{d}{dt}\right)^{k-1}H_n^{\delta,\beta}\left(t\right)\right]_{t=\frac{\pi}{n}}^{t=\pi}=O(1)V_0^{\pi/n}\left(\psi_\alpha\right)$

Proof of Lemma 8 (i)

Using Lemma 4(i) and (5.1.11)

$$\left[\sum_{k=1}^{h}(-1)^k\,\Psi_k\left(t\right)\left(\frac{d}{dt}\right)^{k-1}\tilde{K}_n^{\delta,\beta}\left(t\right)\right]_{t=0}^{t=\pi/n}$$

$$=O(1)\left[\sum_{k=1}^{h}t^k\,|\psi_k\left(t\right)|\,n^k\right]_{t=0}^{t=\pi/n}$$

$$=O(1)\left[\sum_{k=1}^{h}t^kV_0^t\left(\psi_k\right)n^k\right]_{t=0}^{t=\pi/n}$$

$$=O(1)\sum_{k=1}^{h}\left(\frac{\pi^k}{n^k}V_0^t\left(\psi_k\right)n^k\right)$$

$$=O(1)V_0^{\pi/n}\left(\psi_1\right)+O(1)V_0^{\pi/n}\left(\psi_2\right)+...+O(1)V_0^{\pi/n}\left(\psi_h\right)$$

$$=O(1)V_0^{\pi/n}\left(\psi_\alpha\right)$$

Proof of Lemma 8 (ii)

Using Lemma 4(iii) and (5.1.11), we have

$$\left[\sum_{k=1}^{h}(-1)^k\,\Psi_k\left(t\right)\left(\frac{d}{dt}\right)^{k-1}H_n^{\delta,\beta}\left(t\right)\right]_{t=\frac{\pi}{n}}^{t=\pi}$$

$$= O(1)\left[\sum_{k=1}^{h} t^k \,|\psi_k(t)|\, \frac{1}{n^{\delta-k+1} t^{\delta+1}}\right]_{t=\frac{\pi}{n}}^{t=\pi}$$

$$= O(1)\sum_{k=1}^{h} \frac{1}{n^k} V_0^{\pi/n}(\psi_k) \frac{1}{n^{\delta-k+1}} n^{\delta+1}$$

$$= O(1) V_0^{\pi/n}(\psi_k)$$

Hence the Lemma.

4. Proof of the Theorem 1

Following the line of arguments given in (See [5], Vol.1 p.95),we have

$$\tilde{A}_n^{\beta}\left(\tilde{f},x\right) = \frac{-2}{\pi}\int_0^{\pi/n} \psi(t)\tilde{K}_n^{\delta,\beta}(t)\,dt + \frac{2}{\pi}\int_{\pi/n}^{\pi} \psi(t) H_n^{\delta,\beta}(t) \tag{4.1}$$

Now using the inversion formula (See [1])

$$\psi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\, d\Psi_\alpha(u), \qquad 0 \le \alpha < 1 \tag{4.2}$$

in (5.4.1) we get

$$\tilde{A}_n^{\beta}\left(\tilde{f},x\right) = \frac{-2}{\pi}\int_0^{\pi/n}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\, d\Psi_\alpha(u)\right)\tilde{K}_n^{\delta,\beta}(t)\,dt$$

$$+\frac{2}{\pi}\int_{\pi/n}^{\pi}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\, d\Psi_\alpha(u)\right) H_n^{\delta,\beta}(t)\,dt$$

$$= \frac{-2}{\pi}\int_0^{\pi/n}\left(\frac{1}{\Gamma(1-\alpha)}\int_u^{\pi/n} (t-u)^{-\alpha}\, \tilde{K}_n^{\delta,\beta}(t)\,dt\right) d\Psi_\alpha(u)$$

$$+\frac{2}{\pi}\int_{\pi/n}^{\pi}\left(\frac{1}{\Gamma(1-\alpha)}\int_u^{\pi} (t-u)^{-\alpha}\, H_n^{\delta,\beta}(t)\,dt\right) d\Psi_\alpha(u)$$

$$= \frac{-2}{\pi}\int_0^{\pi/n} \tilde{J}(n,u)\, d\Psi_\alpha(u)$$

$$+\frac{2}{\pi}\int_{\pi/n}^{\pi} \tilde{H}(n,u)\, d\Psi_\alpha(u)$$

$$= \frac{-2}{\pi} I_1 + \frac{2}{\pi} I_2 \qquad \text{(say)} \tag{4.3}$$

Now on integration by parts, I_1 reduces to

$$I_1 = \int_0^{\pi/n} \tilde{J}(n,u) d\Psi_\alpha(u)$$

$$= \left[\tilde{J}(n,u) \psi_\alpha(u) \right]_{u=0}^{\pi/n}$$

$$-\int_0^{\pi/n} \frac{u^\alpha \psi_\alpha^{(u)}}{\Gamma(\alpha+1)} \frac{d}{du} \tilde{J}(n,u) du$$

$$= -\frac{1}{\Gamma(\alpha+1)} \int_0^{\pi/n} u^\alpha \psi_\alpha(u) \frac{d}{du} \tilde{J}(n,u) du \tag{4.4}$$

Since

$$\Gamma(1-\alpha) \tilde{J}(n,u) = \int_u^{\pi/n} (t-u)^{-\alpha} \tilde{K}_n^{\delta,\beta}(t) dt$$

$$= \left[\tilde{K}_n^{\delta,\beta}(t) \frac{(t-u)^{-\alpha+1}}{(1-\alpha)} \right]_{t=u}^{\pi/n} - \frac{1}{(1-\alpha)} \int_u^{\pi/n} (t-u)^{-\alpha+1} \left(\frac{d}{dt} \right) \left(\tilde{K}_n^{\delta,\beta}(t) \right) dt$$

$$= \tilde{K}_n^{\delta} \left(\frac{\pi}{n} \right) \frac{\left(\frac{\pi}{n} - u \right)^{1-\alpha}}{(1-\alpha)} - \frac{1}{(1-\alpha)} \int_u^{\pi/n} (t-u)^{1-\alpha} \left(\frac{d}{dt} \right) \left(\tilde{K}_n^{\delta,\beta} \right) dt$$

Hence

$$\Gamma(1-\alpha) \frac{d}{du} \left(\tilde{J}(n,u) \right) = -\left(\frac{\pi}{n} - u \right)^{-\alpha} \tilde{K}_n^{\delta,\beta}(\pi/n)$$

$$+ \int_u^{\pi} (t-u)^{-\alpha} \left(\frac{d}{dt} \right) \left(\tilde{K}_n^{\delta,\beta}(t) \right) dt \tag{4.5}$$

Using the formula (4.5) in I_1 we have

$I_1 = I_{11} + I_{12}$ (say)

By using $|\psi_\alpha(t)| \le V_0^t(\psi_\alpha)$, we have

$$I_{11} = O(1) \int_0^{\pi/n} \left(\frac{\pi}{n} - u \right)^{-\alpha} V_0^u(\psi_\alpha) u^\alpha \left| \left(\tilde{K}_n^{\delta,\beta}(\pi/n) \right) \right| du$$

$$= O(1) n V_0^{\pi/n}(\psi_\alpha) \int_0^{\pi/n} \left(\frac{\pi}{n} - u \right)^{-\alpha} u^\alpha du$$

$$= O(1) n V_0^{\pi/n}\left(\psi_\alpha\right)\frac{\pi^\alpha}{n^\alpha}\left[\frac{\left(\frac{\pi}{n}-u\right)^{-\alpha+1}}{-\alpha+1}\right]_0^{\pi/n}$$

$$= O(1)\frac{V_0^{\pi/n}\left(\psi_\alpha\right)}{n^{\alpha-1}}\left[\frac{0}{-\alpha+1}-\frac{\left(\frac{\pi}{n}\right)^{-\alpha+1}}{-\alpha+1}\right]$$

$$= O(1)\frac{V_0^{\pi/n}\left(\psi_\alpha\right)}{n^{\alpha-1}}\frac{1}{n^{-\alpha+1}}$$

$$= O(1) V_0^{\pi/n}\left(\psi_\alpha\right) \qquad (4.6)$$

In order to evaluate I_{12} we note that

$$\int_u^{\pi/n}(t-u)^{-\alpha}\left(\frac{d}{dt}\right)\left(\tilde{K}_n^{\delta,\beta}(t)\right)dt$$

$$= O(n^2)\int_u^{\pi/n}(t-u)^{-\alpha}\,dt$$

$$= O(n^2)\left(\frac{\pi}{n}-u\right)^{1-\alpha}$$

$$= O(n^2)\frac{1}{n^{1-\alpha}}$$

$$= O(n^{1+\alpha}) \qquad (4.7)$$

by Lemma 4 (i).

Hence

$$I_{12} = O(1) n^{1+\alpha}\int_0^{\pi/n} u^\alpha\,|\psi_\alpha|\,du$$

$$= O(1) n^{1+\alpha}\int_0^{\pi/n} u^\alpha V_0^u\left(\psi_\alpha\right)du$$

$$= O(1) V_0^{\pi/n}\left(\psi_\alpha\right) \qquad (4.8)$$

From (4.6) and (4.8) we have

$$I_1 = O(1)V_0^{\pi/n}\left(\psi_\alpha\right) \tag{4.9}$$

By integration by parts

$$I_2 = \left[\tilde{H}(n,u)\Psi_\alpha(u)\right]_{\pi/n}^{\pi} - \int_{\pi/n}^{\pi}\Psi_\alpha(u)\frac{d}{du}\left(\tilde{H}(n,u)\right)du$$

$$= -\frac{\tilde{H}(n,\pi/n)\psi_\alpha(\pi/n)\pi^\alpha}{\Gamma(\alpha+1)n^\alpha} - \frac{1}{\Gamma(\alpha+1)}\int_{\pi/n}^{\pi}u^\alpha\psi_\alpha(u)\frac{d}{du}\left(\tilde{H}(n,u)\right)du$$

$$= O(1)\frac{|\psi_\alpha(\pi/n)|}{n^{-\alpha}n^\alpha} - \frac{1}{\Gamma(\alpha+1)}\int_{\pi/n}^{\pi}\psi_\alpha(u)\left(u^\alpha\frac{d}{du}\left(\tilde{H}(n,u)\right)\right)du$$

$$= O(1)V_0^{\pi/n}\left(\psi_\alpha\right) + \frac{1}{\Gamma(\alpha+1)}\int_{\pi/n}^{\pi}\psi_\alpha(u)d\tilde{Q}(n,u)du \tag{4.10}$$

by Lemma 5(i)

Now integration by parts

$$\int_{\pi/n}^{\pi}\psi_\alpha(u)d\tilde{Q}(n,u)$$

$$= \left[\psi_\alpha(u)\tilde{Q}(n,u)\right]_{\pi/n}^{\pi} - \int_{\pi/n}^{\pi}d\psi_\alpha(u)\tilde{Q}(n,u)$$

$$= -\psi_\alpha(\pi/n)\tilde{Q}(n,\pi/n) - \int_{\pi/n}^{\pi}d\psi_\alpha(u)\tilde{Q}(n,u)$$

$$= O(1)V_0^{\pi/n}\left(\psi_\alpha\right) - \int_{\pi/n}^{\pi}d\psi_\alpha(u)\tilde{Q}(n,u) \tag{4.11}$$

Since $\left|\psi_\alpha\left(\pi/n\right)\right| \le V_0^{\pi/n}\left(\psi_\alpha\right)$ and $\tilde{Q}\left(n,\pi/n\right) = O(1)$

By Lemma 5(iii) and Lemma 6

$$\int_{\pi/n}^{\pi}d\psi_\alpha(u)\tilde{Q}(n,u) = \frac{O(1)}{n^{\delta+1-\alpha}(\log n)^\beta}\int_{\pi/n}^{\pi}\frac{|d\psi_\alpha(u)|\log^\beta(a\pi/2u)}{u^{\delta+1-\alpha}}$$

$$+\frac{O(1)}{n^{1-\alpha}}\int_{\pi/n}^{\pi}\frac{|d\psi_\alpha(u)|}{u^{1-\alpha}}$$

$$= \frac{O(1)}{n^{\delta+1-\alpha}(\log n)^\beta}\sum_{k=1}^{n}k^{\delta-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right)\log^\beta(ak/2)$$

$$+\frac{O(1)}{n^{1-\alpha}}\sum_{k=1}^{n}k^{-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right) \tag{4.12}$$

From (4.10) (4.11) and (4.12)

$$I_2 = O(1)V_0^{\pi/n}\left(\psi_\alpha\right) + O(1)\frac{1}{n^{\delta+1-\alpha}(\log n)^\beta}\sum_{k=1}^{n}k^{\delta-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right)\log^\beta\left(ak/2\right)$$

$$+\frac{O(1)}{n^{1-\alpha}}\sum_{k=1}^{n}k^{-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right) \tag{4.13}$$

Now collecting the result from (4.9), (4.13) and using in (4.3) we have

$$\tilde{A}_n^{\delta,\beta}\left(\tilde{f},x\right) = O(1)\frac{1}{n^{\delta+1-\alpha}(\log n)^\beta}\sum_{k=1}^{n}k^{\delta-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right)\log^\beta\left(ak/2\right) + \frac{O(1)}{n^{1-\alpha}}\sum_{k=1}^{n}k^{-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right)$$

Proof of the Theorem 2

Following the line of arguments given in (See [5], Vol. 1 p.95) ,we have

$$\tilde{A}_n{}^\beta\left(\tilde{f},x\right) = \frac{-2}{\pi}\int_0^{\pi/n}\psi\left(t\right)\tilde{K}_n^{\delta,\beta}\left(t\right)dt$$

$$+\frac{2}{\pi}\int_{\pi/n}^{\pi}\psi\left(t\right)H_n^{\delta,\beta}\left(t\right)dt$$

For h=[α] and α≥1

Now repeated integration by parts h=[α] times yields

$$\tilde{A}_n{}^\beta\left(\tilde{f},x\right) = \frac{-2}{\pi}\left[\sum_{k=1}^{h}(-1)^k\,\Psi_k\left(t\right)\left(\frac{d}{dt}\right)^{k-1}\tilde{K}_n^{\delta,\beta}\left(t\right)\right]_{t=0}^{\pi/n}$$

$$+\frac{2}{\pi}\left[\sum_{k=1}^{h}(-1)^k\,\Psi_k\left(t\right)\left(\frac{d}{dt}\right)^{k-1}H_n^{\delta,\beta}\left(t\right)\right]_{t=\pi/n}^{\pi}$$

$$+\frac{2}{\pi}(-1)^{h+1}\int_0^{\pi/n}\Psi_h\left(t\right)\left(\frac{d}{dt}\right)^h\tilde{K}_n^{\delta,\beta}\left(t\right)+\frac{2}{\pi}(-1)^h\int_{\pi/n}^{\pi}\Psi_h\left(t\right)\left(\frac{d}{dt}\right)^h H_n^{\delta,\beta}\left(t\right)$$

$$= O(1)V_0^{\pi/n}\left(\psi_\alpha\right) + O(1)V_0^{\pi/n}\left(\psi_\alpha\right)$$

$$+\frac{2}{\pi}(-1)^{h+1}\int_0^{\pi/n}\Psi_h\left(t\right)\left(\frac{d}{dt}\right)^h K_n^{\delta,\beta}\left(t\right)dt$$

$$+\frac{2}{\pi}(-1)^{h}\int_{\pi/n}^{\pi}\Psi_h\left(t\right)\left(\frac{d}{dt}\right)^h H_n^{\delta,\beta}\left(t\right)dt$$

$$= O(1)V_0^{\pi/n}\left(\psi_\alpha\right) + \frac{2}{\pi}(-1)^{h+1}K_1 + \frac{2}{\pi}(-1)^h K_2 \qquad \text{(Say)} \tag{4.14}$$

by Lemma 8(i) and (ii).

Where $K_1 = \int_0^{\pi/n} \Psi_h(t)\left(\frac{d}{dt}\right)^h K_n^{\delta,\beta}(t)\,dt$ (4.15)

Now using the inversion formula (See [1])

$$\Psi_h(t) = \frac{1}{\Gamma(1+h-\alpha)}\int_0^t (t-u)^{h-\alpha}\, d\Psi_\alpha(u) \qquad \alpha\geq 1 \tag{4.16}$$

in (5.4.15) we get

$$K_1 = \int_0^{\pi/n}\left(\frac{1}{\Gamma(1+h-\alpha)}\int_0^t (t-u)^{h-\alpha}\, d\Psi_\alpha(u)\right)\left(\frac{d}{dt}\right)^h \tilde{K}_n^{\delta,\beta}(t)\,dt$$

$$= \int_0^{\pi/n}\left(\frac{1}{\Gamma(1+h-\alpha)}\int_u^{\pi/n} (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^h \tilde{K}_n^{\delta,\beta}(t)\,dt\right) d\Psi_\alpha(u)$$

$$= \int_0^{\pi/n} \tilde{J}_1(n,u)\, d\Psi_\alpha(u)$$

$$= \left[\tilde{J}_1(n,u)\Psi_\alpha(u)\right]_{t=0}^{\pi/n} - \int_0^{\pi/n} \frac{u^\alpha \psi_\alpha(u)}{\Gamma(\alpha+1)}\frac{d}{du}\left(\tilde{J}_1(n,u)\right)du$$

$$= \frac{-1}{\Gamma(\alpha+1)}\int_0^{\pi/n} u^\alpha \psi_\alpha(u)\frac{d}{du}\left(\tilde{J}_1(n,u)\right)du \tag{4.17}$$

Since

$$\Gamma(1+h-\alpha)\tilde{J}_1(n,u) = \int_u^{\pi/n} (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^h \tilde{K}_n^{\delta,\beta}(t)\,dt$$

$$= \left[\left(\frac{d}{dt}\right)^h \tilde{K}_n^{\delta,\beta}(t)\frac{(t-u)^{1+h-\alpha}}{1+h-\alpha}\right]_{t=u}^{\pi/n}$$

$$-\frac{1}{(1+h-\alpha)}\int_u^{\pi/n} (t-u)^{1+h-\alpha}\left(\frac{d}{dt}\right)^{h-1} \tilde{K}_n^{\delta,\beta}(t)\,dt$$

$$= \frac{\left(\frac{\pi}{n}-u\right)^{1+h-\alpha}\left[\left(\frac{d}{dt}\right)^h \tilde{K}_n^{\delta,\beta}(t)\right]_{t=\pi/n}}{(1+h-\alpha)}$$

$$-\frac{1}{(1+h-\alpha)}\int_u^{\pi/n}(t-u)^{1+h-\alpha}\left(\frac{d}{dt}\right)^{h+1}\tilde{K}_n^{\delta,\beta}(t)\,dt$$

$$\Gamma(1+h-\alpha)\frac{d}{du}\left(\tilde{J}_1(n,u)\right)=-\left(\frac{\pi}{n}-u\right)^{h-\alpha}\left[\left(\frac{d}{dt}\right)^{h}\tilde{K}_n^{\delta,\beta}(t)\right]_{t=\pi/n}$$

$$+\int_u^{\pi/n}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}\tilde{K}_n^{\delta,\beta}(t)dt \qquad (4.18)$$

Using the formula (4.18) in (4.17) we obtain

$K_1 = K_{11} + K_{12}$

By using $\left|\psi_\alpha(t)\right|\le V_0^t(\psi_\alpha)$

$$K_{11}=O(1)n^{h+1}\int_0^{\pi/n}u^\alpha\left|\psi_\alpha(u)\right|\left(\frac{\pi}{n}-u\right)^{h-\alpha}du$$

$$=O(1)n^{h+1}\int_0^{\pi/n}u^\alpha V_0^u(\psi_\alpha)\left(\frac{\pi}{n}-u\right)^{h-\alpha}du$$

$$=O(1)n^{h+1}\frac{1}{n^\alpha}V_0^{\pi/n}(\psi_\alpha)\int_0^{\pi/n}\left(\frac{\pi}{n}-u\right)^{h-\alpha}du$$

$$=O(1)\frac{1}{n^{\alpha-h-1}}V_0^{\pi/n}(\psi_\alpha)\left[\frac{\left(\frac{\pi}{n}-u\right)^{1+h-\alpha}}{1+h-\alpha}\right]_0^{\pi/n}$$

$$=O(1)\frac{1}{n^{\alpha-h-1}}V_0^{\pi/n}(\psi_\alpha)\frac{1}{n^{1+h-\alpha}}$$

$$=O(1)V_0^{\pi/n}(\psi_\alpha) \qquad (4.19)$$

For estimate of K_{12} we note that

$$\int_u^{\pi/n}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}\tilde{K}_n^{\delta}(t)\,dt$$

$$=O(n^{h+2})\int_u^{\pi/n}(t-u)^{h-\alpha}\,dt$$

$$= O(n^{h+2})\left(\pi/_n - u\right)^{1+h-\alpha}$$

$$= O(n^{h+2})\frac{1}{n^{1+h-\alpha}}$$

$$= O(n^{1+\alpha})$$

Thus

$$K_{12} = O(n^{\alpha+1})\int_0^{\pi/n} u^{\alpha}\left|\psi_{\alpha}(u)\right| du$$

$$= O(n^{\alpha+1})\int_0^{\pi/n} u^{\alpha} V_0^{u}(\psi_{\alpha})\, du$$

$$= O(n^{\alpha+1}) V_0^{\pi/n}\left[\frac{u^{\alpha+1}}{\alpha+1}\right]_0^{\pi/n}$$

$$= O(1) V_0^{\pi/n}(\psi_{\alpha}) \tag{4.20}$$

From (4.19) & (4.20)

$$K_1 = K_{11} + K_{12} = O(1) V_0^{\pi/n}(\psi_{\alpha}) \tag{4.21}$$

$$K_2 = \int_{\pi/n}^{\pi} \Psi_h(t)\left(\frac{d}{dt}\right)^h H_n^{\delta,\beta}(t)\, dt \tag{4.22}$$

Now using the inversion formula (See (1))

$$\Psi_h(t) = \frac{1}{\Gamma(1+h-\alpha)}\int_0^t (t-u)^{h-\alpha}\, d\Psi_{\alpha}(u) \qquad \alpha\geq 1 \tag{4.23}$$

in (5.4.22)

$$K_2 = \int_{\pi/n}^{\pi}\left(\frac{1}{\Gamma(1+h-\alpha)}\int_0^t (t-u)^{h-\alpha}\, d\Psi_{\alpha}(u)\right)\left(\frac{d}{dt}\right)^h H_n^{\delta,\beta}(t)\, dt$$

$$= \int_{\pi/n}^{\pi}\left(\frac{1}{\Gamma(1+h-\alpha)}\int_u^{\pi} (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^h H_n^{\delta,\beta}(t)\, dt\right) d\Psi_{\alpha}(u)$$

$$= \int_{\pi/n}^{\pi} \tilde{H}_1(n,u)\, d\Psi_{\alpha}(u)$$

$$=\left[\tilde{H}_1(n,u)\Psi_\alpha\left(u\right)\right]_{t=\pi/n}^{\pi}-\int_{\pi/n}^{\pi}\frac{u^\alpha\psi_\alpha\left(u\right)}{\Gamma\left(\alpha+1\right)}\frac{d}{du}\left(\tilde{H}_1(n,u)\right)du$$

$$=-\frac{\tilde{H}_1(n,\pi/n)\psi_\alpha\left(\pi/n\right)\pi^\alpha}{\Gamma\left(\alpha+1\right)n^\alpha}-\frac{1}{\Gamma\left(\alpha+1\right)}\int_{\pi/n}^{\pi}u^\alpha\psi_\alpha\left(u\right)\frac{d}{du}\tilde{H}_1(n,u)du$$

$$=O(1)n^\alpha\frac{V_0^{\pi/n}\left(\psi_\alpha\right)}{n^\alpha}-\frac{1}{\Gamma\left(1+\alpha\right)}\int_{\pi/n}^{\pi}\psi_\alpha\left(u\right)\left(u^\alpha\frac{d}{du}\tilde{H}_1(n,u)\right)du$$

$$=O(1)V_0^{\pi/n}\left(\psi_\alpha\right)+\frac{1}{\Gamma\left(1+\alpha\right)}\int_{\pi/n}^{\pi}\psi_\alpha\left(u\right)d\tilde{Q}_1(n,u)du \tag{4.24}$$

by **Lemma 5**

Now integrating by parts

$$\int_{\pi/n}^{\pi}\psi_\alpha\left(u\right)d\tilde{Q}_1(n,u)du$$

$$=\left[\psi_\alpha\left(u\right)\tilde{Q}_1(n,u)\right]_{\pi/n}^{\pi}-\int_{\pi/n}^{\pi}d\psi_\alpha\left(u\right)\tilde{Q}_1(n,u)du$$

$$=-\psi_\alpha\left(\pi/n\right)\tilde{Q}_1(n,\pi/n)-\int_{\pi/n}^{\pi}d\psi_\alpha\left(u\right)\tilde{Q}_1(n,u)du$$

$$=O(1)V_0^{\pi/n}\left(\psi_\alpha\right)-\int_{\pi/n}^{\pi}d\psi_\alpha\left(u\right)\tilde{Q}_1(n,u)du \tag{4.25}$$

Since $\left|\psi_\alpha\left(\pi/n\right)\right|\le V_0^{\pi/n}\left(\psi_\alpha\right)$ and $\tilde{Q}_1(n,\frac{\pi}{n})=O(1)$

By Lemma 5(iv).

Further by Lemma 5(iv) and Lemma 7(iii) & 8(iv)

$$\int_{\pi/n}^{\pi}d\psi_\alpha\left(u\right)\tilde{Q}_1(n,u)=$$

$$\frac{O(1)}{(\log n)^\beta n^{\delta+1-\alpha}}\int_{\pi/n}^{\pi}\frac{\log^\beta\left(d/u\right)\left|d\Psi_\alpha(u)\right|}{u^{\delta+1-\alpha}}+\frac{O(1)}{(\log n)^\beta n^{\delta-\alpha}}\int_{\pi/n}^{\pi}\frac{\log^\beta\left(d/u\right)\left|d\Psi_\alpha(u)\right|}{u^{\delta-\alpha}}$$

$$=\frac{O(1)}{(\log n)^\beta n^{\delta+1-\alpha}}\sum_{1}k^{\delta-\alpha}V_0^{\pi/k}\left(\psi_\alpha\right)\log^\beta k$$

$$+\frac{O(1)}{(\log n)^{\beta}\, n^{\delta-\alpha}}\sum_{k=1}^{n} k^{\delta-\alpha-1} V_0^{\pi/k}\left(\psi_\alpha\right)\log^{\beta} k \qquad (4.26)$$

From (4.24) (4.25) and (4.26)

$$K_2 = O(1)V_0^{\pi/n}\left(\psi_\alpha\right) + \frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n} k^{\delta-\alpha} V_0^{\pi/k}\left(\psi_\alpha\right)\ \log^{\beta} k$$

$$+\frac{O(1)}{n^{\delta-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n} k^{\delta-\alpha-1} V_0^{\pi/k}\left(\psi_\alpha\right)\ \log^{\beta} k \qquad (4.27)$$

Now collecting the results from (4.21) (4.27) and using in (4.14) we get

$$\tilde{A}_n^{\beta}\left(\tilde{f},x\right) = O(1)V_0^{\pi/n}\left(\psi_\alpha\right) + \cdots + \frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n} k^{\delta-\alpha} V_0^{\pi/k}\left(\psi_\alpha\right)\log^{\beta} k$$

$$+\frac{O(1)}{n^{\delta-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n} k^{\delta-\alpha-1} V_0^{\pi/k}\left(\psi_\alpha\right)\log^{\beta} k$$

Further using Lemma 7(i)

$$\tilde{A}_n^{\delta,\beta}\left(f,x\right) = \frac{O(1)}{n^{\delta+1-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n} k^{\delta-\alpha} V_0^{\pi/k}\left(\psi_\alpha\right)\log^{\beta} k + \frac{O(1)}{n^{\delta-\alpha}(\log n)^{\beta}}\sum_{k=1}^{n} k^{\delta-\alpha-1} V_0^{\pi/k}\left(\psi_\alpha\right)\log^{\beta} k$$

Hence the proof of the Theorem 2

REFERENCES

[1] Bosanquet, L.S.: "Notes on absolute Summability of a Fourier series", Proc. Lond. Math. Soc. XI, (1936)

[2] Chandra, P. and Dikshit, G.D.: "The absolute generalized harmonic-Cesaro summability of a Fourier series", Analysis Mathematical, 20 (1994), 173-183.

[3] Das, G. and Mohapatra, P.C.: "On a generalized Harmonic-Cesaro method of summation and its application to Fourier series", Indian Journal of Mathematics, Vol. 22, No.1, January 1980, 33-37, Proc. C. T. Rajgopal Memorial Volume.

[4] Mazhar, S.M. and AL-Budaiwi, A.: "An estimate of the rate of convergence of the conjugate Fourier series of functions of bounded variation", Acta Mathematica Hungarica 49, 1987.

[5] Zygmund, A.: "Trigonometric series, second edition", Volume I and Volume II combined, Cambridge University Press, 1933.

[6] Behera, S., Degree of Approximation of Conjugate Fourier Series of Function of Bounded Variation by $\left(z,\delta,\beta\right)$ method, International Journal of Mathematical Achieve, 5(5), 2014, 1-11

[7] Behera, S., Degree of Approximation of Fourier Series of Fractional Mean of Function of Bounded Variation by $\left(z,\delta,\beta\right)$ method, Journal of Advances in Mathematics, 9(1), 2014,1684-1703.

AUTHOR'S BIOGRAPHY

Dr. Samira Behera has been teaching in the Department of Mathematics since July, 1998. He was awarded M.Phil. and Ph.D. from Utkal University, Odisha.