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Lie Triple Derivations of Algebras of Measurable Operators

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Abstract: *We prove that every Lie triple derivation on algebras of measurable operators is in standard form, that is, it can be uniquely decomposed into the sum of a derivation and a center-valued trace.*

Keywords: *von Neumann algebras, measurable operator, type I von Neumann algebras, derivation, inner derivation, Lie triple derivation, center-valued trace.*

1. INTRODUCTION

linear operator $D: A \to A$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations*. If the element a implementing the derivation D_a on A, belongs to a larger algebra B, containing A (as a proper ideal as usual) then D_a is called a *spatial derivation*.

A linear operator L:$A \to A$ is called a *Lie triple derivation* if

$L[[x,y],z]=[[L(x),y],z]+[[x,L(y)],z]+[[x,y],L(z)]]$, for all $x,y,z \in A$, where $[x,y]=xy-yx$.

Denote by $Z(A)$ the center of A.

A linear operator $\tau: A \to Z(A)$ is called *a center-valued trace* if $\tau(xy)=\tau(yx)$, $\forall x,y \in A$.

Let H be a Hilbert space, $B(H)$ be the algebra of all bounded linear operators acting in H, M be a von Neumann subalgebra in $B(H)$, $P(M)$ be a complete lattice of all orthoprojections in M.

A linear subspace $\mathcal{D}$ on H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subseteq \mathcal{D}$ for every unitary operator u from the commutant $M' = \{ y \in B(H) : xy = yx, \forall x \in M \}$ of the algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$), if $u(\mathcal{D}(x)) \subseteq \mathcal{D}(x)$ and $ux(\xi)=xu(\xi)$ for every unitary operator $u \in M'$, and all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dens* in H with respect to the von Neumann algebra M, if

1) $\mathcal{D}\eta M$,
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty} \subset P(M)$, such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$, and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity M.

A closed linear operator x, on a H, is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$, and $\mathcal{D}(x)$ is strongly dens in H. Denote by $S(M)$ the set of all measurable operators affiliated with *M* (see. [5,13]) and the center of an algebra *S*(*M*) by *Z*(*S*(*M*)). A von Neumann algebra *M* is of type *I* if it contains a faithful abelian projection *e* (i.e. *eMe* is an abelian(commutative) von Neumann algebra).

If p_i, p_j are projectors in $S(M)$, then $p_iS(M)p_j = p_iAp_j : A \in S(M)$, $i, j = 1,2.$ Set $p_1 = p$ and $p_2 = 1 - p.$ Then $S(M) = \sum_{i=1}^{2}\sum_{j=1}^{2} p_iS(M)p_j.$ Let further $M_{ij} = p_iS(M)p_j$, $i, j = 1,2.$ Recall that $M_{ij} = M_{ik}M_{kj}$, for $i, j = 1,2$.

2. Results and Discussion

Let $L : S(M) \to S(M)$ *be Lie triple derivation.*

Lemma1. *If* $[x,y] \in Z(S(M))$ *for* $x,y \in S(M)$*, then*

$$[L(x),y]+[x,L(y)] \in Z(S(M)).$$

Proof. $0 = L(0) = L[[x, y], z] = [[L(x), y], z] + [[x, L(y)], z] = [[L(x), y] + [x, L(y)], z]$ for all $z \in S(M)$.

Lemma 2. *For any projector* $p \in S(M)$*,*

$$pL(p)px+xpL(p)p=\{L(p)-L(p)p-pL(p)+2pL(p)p\}xp+$$

$$+px\,\{L(p)-L(p)p-pL(p)+2pL(p)p\}.$$

Proof. Applying L to the identity $[[[[x, p], p], p]\,[[[[x, p], p], p], p] = [[x, p], p]$ we obtained the required equality.

Lemma 3. $L(p_1) = [p_{1,}s] + z$, *where* $z \in Z(S(M))$, $s \in S(M)$.

Proof. Let $L(p_1) = \sum e_{ij}$, $e_{ij} \in M_{ij}$ (i,j=1,2).

Applying Lemma 2 for all $x \in S(M)$, we obtain

$e_{11}x + xe_{11} = (e_{11} + e_{22})xp + px(e_{11} + e_{22})$. If $x \in M_{12}$, then, $e_{11}x = xe_{22}$, what follows $(e_{11} + e_{22})x = x(e_{11} + e_{22}) \quad (x \in M_{12})$.

Analogously, $(e_{11} + e_{22})x = x(e_{11} + e_{22}) \quad (x \in M_{21})$. Let now $x \in M_{11}$ and $y \in M_{12}$. Then $\{(e_{11} + e_{22})x - x(e_{11} + e_{22})\}y = (e_{11} + e_{22})xy - xy(e_{11} + e_{22}) =$
$(e_{11} + e_{22})xy - (e_{11} + e_{22})xy = 0,$

Since $y, xy \in M_{12}$. It follows that $(e_{11} + e_{22})x - x(e_{11} + e_{22}) = 0 \quad (x \in M_{11})$.

Similarly $(e_{11} + e_{22})x - x(e_{11} + e_{22}) = 0 \quad (x \in M_{22})$, i.e. $e_{11} + e_{22} = z \in Z(S(M))$. Since $L(p_1) = (e_{12} + e_{21}) + z$ and, setting $s = e_{12} - e_{21}$, we obtain $L(p_1) = (p_1s - sp_1) + z$.

Following from this lemma, we can put $L(p_1) \in Z(S(M))$. For, if the theorem is proved with this restriction, the general theorem can be proved by looking at *L*'(*x*)=*L*(*x*)−[*x*,*s*].

Lemma 4. *If* $x \in M_{ij}$ $i \neq j$*, then* $L(x) \in M_{ij}$.

Proof. $x \in M_{12}$, $x = [[x, p_1], p_1]$. Let $L(x) = \sum_{1\le i,j\le 2} x_{ij}$, where $x_{ij} = p_iL(x)p_j$. then $\sum_{1\le i,j\le 2} x_{ij} = L(x) = [[L(x), p_1], p_1] = x_{12} + x_{21}$. If $x, y \in M_{12}$, then $[x, y] = 0$, therefore, by Lemma 1, $c = [L(x), y] + [x, L(y)] \in Z(S(M))$. Since $x = [p_{1,}x]$, we have

$[L(x), y] = [L[p_1, x], y] = c - [[p_1, x], L(y)] = c - L[[p_1, x], y] + [[L(p_1), x], y] +$ $[[p_1, L(x)], y] = c + [[p_1, L(x)], y],$ what implies

$[x_{12} + x_{21}, y] = c + [[p_1, x_{12} + x_{21}], y] = c + [x_{12} - x_{21}, y]$, hence $[x_{21}, y] = \frac{1}{2}c \in Z(S(M))$. We conclude, that $x_{21}y - yx_{21} = 0$ for all $y \in M_{12}$.

Thus, $x_{21}y = 0$ for all $y \in M_{12}$, hence, $x_{21} = 0$. The case of $x \in M_{21}$ can be proved analogously.

Lemma 5. *If* $x \in M_{ii}$, *then* $L(x) \in M_{ii} + Z(S(M))$.

Proof. If $x \in M_{11}$, we have $0 = [[x, p_1], p_1]$, that is why $0 = [[L(x), p], p_1] = x_{21} + x_{12}$. Hence, $x_{12} = x_{21} = 0$. Thus, $L(x) = x_{11} + x_{22} \in M_{11} + M_{22}$. Let $x \in M_{11}, y \in M_{22}$. Then $0 = [x, y]$, therefore $[L(x), y] + [x, L(y)] \in Z(S(M))$. Let $L(x) = x_{11} + x_{22}$, $L(y) = y_{11} + y_{22}$. Then $x, y \in M_{ij}$ $[x_{11} + x_{22}, y] + [x, y_{11} + y_{22}] = [x_{22}, y] + [x, y_{11}] = z \in Z(S(M))$. It follows that $x_{22} \in Z(M_{22})$. Thus $x_{22} = cp_2 = c(1 - p_1) \in M_{11} + Z(S(M))$.

Hence $L(x) = x_{11} + x_{22} = x_{11} - cp_1 + c \in M_{11} + Z(S(M))$.

Definition. If $x \in M_{ij}$, $i \neq j$, suppose $D(x) = L(x)$. If $x \in M_{ij}, i = j$, then $L(x) = x' + z$, where $x' \in M_{ij}$, $z \in Z(S(M))$. In this case $D(x) = x'$. Defining in this way D on the all $S(M)$, we put $\tau(x) = L(x) - D(x)$.

Lemma 6. *The mapping τ:S(M)→Z(S(M)) is a linear mapping.*

Proof. Homogeneity of τ is obvious. Let us show additivity of it. Let $x, y \in M_{ij}$. Then we have

$$\tau(x + y) - \tau(x) - \tau(y) = L(x + y) - D(x + y) - L(x) + D(x) - L(y) + D(y) =$$
$$[D(x) + D(y) - D(x + y)] \in M_{11} \cap Z(S(M)) = 0.$$

Lemma 7. *If* $x \in M_{ii}, y \in M_{jk}$ *(j≠k), then D(xy)=D(x)y+xD(y).*

Proof. If $i \neq j$, then $xy = 0$. $D(x)y = 0$ and $xD(y) = 0$. If $i = j, x \in M_{11}, y \in M_{12}$, then $xy \in M_{12}$ and $D(xy) = L(xy)$, since $xy = [x, y] = -[[p_1, x], y]$. Hence

$$D(xy) = -L[[p_1, y], x] = -[[p_1, L(y)], x] - [[p_1, y], L(x)] = -[L(y), x] -$$
$$[y, L(x)] = [x, L(y)] + [L(x), y] = [x, D(y)] + [D(x), y] = xD(y) + D(x)y.$$

Lemma 8. *If* $x \in M_{ii}, y \in M_{jj}$, *then D(xy)=xD(y)+D(x)y.*

Proof. Let $x, y \in M_{11}$. *For* $r \in M_{12}$, *by Lemma 7, we obtain*

$$D(xy)r = D(xyr) - xyD(r) = D(x)yr + xD(yr) - xyD(r) =$$
$$D(x)yr + x\{D(y)r + yD(r)\} - xyD(r)\} = \{D(x)y + xD(y)\}r.$$

Since $\{D(xy) - D(x)y - xD(y)\}r = 0$ *for all* $r \in M_{12}$. *It follows that*

$$D(xy) - D(x0y - xD(y) = 0.$$

Lemma 9 *D(xyx)=D(x)yx+xD(y)x+xyD(x) for every x∈*M_{ij} *(i≠j) and y∈S(M).*

Proof. Let $x \in M_{ij}(i \neq j)$, $2xyx = [[x, y], x]$. Then

$$2D(xyx) = L(2xyx) = L([[x, y], x]) = [[L(x), y] + [x, L(y)], x] + [[x, y], L(x)] =$$
$$[[D(x), y] + [x, D(y), x] + [[x, y], D(x)] = 2\{D(x)yx + xD(y)x + xyD(x)\}$$

Lemma 10. *The mapping D is an associated derivation on S(M).*

Proof. It is sufficient to show the equality $D(xy) = D(x)y + xD(y)$ for the case of $x \in M_{12}, y \in M_{21}$. We have

$$\tau[x, y] = D(x)y + xD(y) - D(xy) + D(yx) - D(y)x - yD(x) = z \in Z(S(M)) \quad (1)$$

Muiltiplying the equality (1) from the left on x and on y, respectively, we obtain

$$xD(yx) - xD(y)x - xyD(x) = xz \qquad (2)$$
$$yD(x)y + yxD(y) - yD(xy) = yz \qquad (3)$$

It is clear, $yx \in M_{22}$, $x \in M_{12}$, therefore by Lemma 7 $D(xyx) = D(x0yx + xD(yx)$. Sing the equality (2) and Lemma 9, we obtain

$$0 = D(xyx) - D(x)yx - xD(y)x - xyD(x) = xz .$$

Similarly, using the equality (3), we obtain $yz = 0$.

$xz = 0$ implies

$|x|z = 0$ and therefore $|x|z^* = 0$. Hence $xz^* = v|x|z^* = 0,$ where $x = v|x|$ is the polar decomposition of x. We obtain similarly $yz^* = 0$. Multiplying (1) on z^*, we obtain $(D(yx) - D(xy))z^* = zz^*$. We have $D(yx)z^* = D(yx)p_2z^* = D(yxp_2z^*) - yxD(p_2z^*) = -(yxD(p_2z^*))$.

Similarly, $D(xy)z^* = -xyD(p_1z)$. Hence

$$zz^* = (D(yx) - D(xy))z^* = xyD(p_1z^*) - yxD(p_2z^*) .$$

Thus $z^*zz^* = 0$, what implies $z = 0$. It follows from the equality (1)

$D(x)y + xD(y) - D(xy) = -D(yx) + D(y)x + yD(x) = 0$, since $x \in M_{12}, y \in M_{21}$.

Corollary. $\tau[x, y] = 0$ for all $x, y \in S(M)$.

Now we can formulate the main theorem.

Theorem 1. *Let L:S(M)→S(M) be a Lie triple derivation . Then L=D+τ, where D is an associated derivation and τ is a center-valued trace from S(M) into Z(S(M)).*

Let A be a commutative algebra and let $M_n(A)$ be the algebra of $n\times n$ matrices over A. If e_{ij} $i, j = 1,2,...,n$ are the matrix units in $M_n(A)$, then each element $x \in M_n(A)$, has the form

$$x = \sum_{i,j=1}^{n} \lambda_{ij} e_j, \lambda_{ij} \in A, i, j = 1,2,...,n$$

Let $\delta: A\to A$, be a derivation. Setting

$$D_\delta(\sum_{i,j=1}^{n} \lambda_{ij} e_{ij}) = \sum_{i,j=1}^{n} \delta(\lambda_{ij}) e_{ij} \qquad (4)$$

we obtain a well-defined linear operator D_δ on the algebra $M_n(A)$. Moreover D_δ is a derivation on the algebra $M_n(A)$ and its restriction onto the center of the algebra $M_n(A)$ coincides with the given δ. Now Lemma 2.2 [1] implies the following

Corollary. Let M be a homogenous von Neumann algebra of type $I_n, n \in N$. Every Lie triple derivation L on the algebra *S(M)* can be uniquely represented by as a sum $L = D_a + D_\delta + \tau$, where D_a is an inner derivation implemented by an element $a \in S(M)$ while, D_δ is the derivation of the form (4) genereted by a derivation δ on the center $S(M)$ identifed with S(Z).

Now let *M* be an arbitrary finite von Neumann algebra of type *I* with the center *Z*. There exists a family $\{z_n\}_{n\in F}, F \subseteq N$, of central projections from *M* with $\sup_{n\in F} z_n = 1$ such that the algebra *M* is $*$-isomorphic with the C^* –product of von Neumann algebras $z_n M$ of type I_n, respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_n M$$

By Proposition 1.1 [1] we have that

$$S(M) \cong \prod_{n\in F} S(z_n M).$$

Suppose that *D* is a derivation on *S*(*M*), and δ is its restriction onto its center *S*(*Z*). Since δ maps each $z_n S(Z) \cong Z(S(z_n M))$ into itself, δ generates a derivation δ_n on $z_n S(Z)$ for each $n \in F$. Let D_{δ_n} be the derivation on the matrix algebra $M_n(z_n Z(S(M))) \cong S(z_n M)$ defined as in (4). Put

$$D_\delta(\{x_n\}_{n\in F}) = \{D_{\delta_n}(x_n)\}, \{x_n\}_{n\in F} \in S(M). \qquad (5)$$

Then the map D_δ is a derivation on *S*(*M*). Now Lemma 2.3 [1] implies the following

Corollary. Let M be a finite von Neumann algebra of type I. Every Lie triple derivation L on the algebra $S(M)$ can be uniquely represented as a sum $L = D_a + D_\delta + \tau$ where D_a is an inner derivation implemented by an element $a \in S(M)$, and D_δ is a derivation given as (5)

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