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Some Curious Polynomial Expressions of the Trigonometric Functions

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Abstract: *It is noted that cos (4n+1)x and cos (4n-1)x are polynomials of degree 4n+1 and 4n-1 respectivelyin cos x only.It is worth noting that sin (4n+1)xis same (4n+1)th degree polynomial as cos (4n+1)x with cos xreplaced by sin x. In contrast to this sin(4n-1)x is the same (4n-1)th degree polynomial in sin x with sign reversed .*

i.e.IfCos (4n+1)x = f_{4n+1}(cos x) then Sin (4n +1)x = f_{4n+1} (sin x)

Cos (4n-1)x = g_{4n-1}(cos x) then Sin(4n-1)x = - g_{4n-1}(sin x).

The results are proved in two ways: Employing De movire's and Binomial theorems and by using the properties of trigonometric ratios of complimentary angles.

Keywords: *Trigonometric polynomials,sin (nx), cos (nx).*

INTRODUCTION

For 'n' a positive integer, *cos(nx)*and *sin(nx)* can be expressed as polynomials in *sinx* and *cosx* using Demoivre's and Binomial theorems. In this note a theorem on these polynomials is established.

Theorem Let n be a positive integer,

If ***Cos(4n+1)x=f_{4n+1}(cosx)*****then** ***Sin (4n+1)x =f_{4n+1} (sin x)*** **(I)**

If ***Cos (4n-1)x = g_{4n-1}(cosx)*****then** ***Sin (4n-1)x = - g_{4n-1}(sin x)*** **(II)**

wheref_{4n+1} () and g_{4n-1}( ) are polynomials of degrees (4n+1) and (4n-1) respectively.

The results (I) and (II) are established in two diffent ways.

(I) Proof employing De moire's and Binomials:-

When n is a positive integer ,by De moivre's theorem

$Cosnx+i\, Sinnx = e^{inx}=(Cos\, x + i\, Sin\, x)^n$, where $i = \sqrt{-1}$

Consider

$$Cos(4n+1)\, x+iSin(4n+1)x = e^{i(4n+1)x}=(Cosx+iSinx)^{4n+1} \quad (1)$$

Expanding the R.H.S of (1) employing Binomial theorem

Cos (4n+1)x + i Sin (4n+1)x

$$=\begin{Bmatrix} (\cos^{4n+1} x)-\binom{4n+1}{2}(\cos^{4n-1} x)(\sin^2 x)+\binom{4n+1}{4}(\cos^{4n-3} x)(\sin^4 x)-\binom{4n+1}{6}(\cos^{4n-5} x)(\sin^6 x)+..... \\ +\binom{4n+1}{4n}\cos x(\sin^{4n} x) \end{Bmatrix}+i$$

$$\left\{\begin{array}{l}\binom{4n+1}{1}(\cos^{4n}x)\sin x-\binom{4n+1}{3}(\cos^{4n-2}x)(\sin^3 x)+\binom{4n+1}{5}(\cos^{4n-4}x)(\sin^5 x)-\ldots\ldots\ldots\ldots\ldots\\+\binom{4n+1}{4n+1}(\sin^{4n-1}x)\end{array}\right\} \quad (2)$$

Equating real and imaginary parts on both sides of (2) , we get

Cos(4n+1)=

$$\left\{(\cos^{4n+1}x)-\binom{4n+1}{2}(\cos^{4n-1}x)(\sin^2 x)+\binom{4n+1}{4}(\cos^{4n-3}x)(\sin^4 x)-\binom{4n+1}{6}(\cos^{4n-5}x)(\sin^6 x)+\ldots..+\binom{4n+1}{4n}\cos x(\sin^{4n}x)\right\}$$

(3)

and

$$Sin(4n+1)x=\left\{\begin{array}{l}\binom{4n+1}{1}(\cos^{4n}x)(\sin x)-\binom{4n+1}{3}(\cos^{4n-2}x)(\sin^3 x)+\ldots\ldots..\\\binom{4n+1}{4n-1}(\cos^2 x)(\sin^{4n-1}x)+(\sin^{4n+1}x)\end{array}\right\} \quad (4)$$

Now the expression on the R.H.S of (3) can be written as

$$Cos\ (4n+1)x=\begin{array}{l}(\cos^{4n+1}x)-\binom{4n+1}{2}(\cos^{4n-1}x)(1-\cos^2 x)+\binom{4n+1}{4}(\cos^{4n-3}x)(1-\cos^2 x)^2-\\\binom{4n+1}{6}(\cos^{4n-5}x)(1-\cos^2 x)^3\end{array}$$

$$\ldots\ldots.+(-1)^p\binom{4n+1}{2p}(\cos^{4n-2p+1}x)(1-\cos^2 x)^p+\ldots\ldots\ldots\ldots\ldots\ldots\ldots.-\binom{4n+1}{4n}(\cos x)(1-\cos^2 x)^{2n}$$

$$=\sum_{p=0}^{2n}(-1)^p\binom{4n+1}{2p}(\cos x)^{4n-2p+1}(1-\cos^2 x)^p = f_{4n+1}(\cos x)\ldots.(say) \quad (5)$$

Also, writing the expression on the R.H.S of expression (4) in the reverse order, we get

Sin (4n+1)x =

$$(\sin^{4n+1}x)-\binom{4n+1}{4n-1}(1-\sin^2 x)(\sin^{4n-1}x)+\ldots.+(-1)^p\binom{4n+1}{2p+1}(1-\sin^2 x)^{2n-p}(\sin^{2p+1}x)$$

$$\ldots\ldots\ldots\ldots.-\binom{4n+1}{3}(1-\sin^2 x)^{2n-1}(\sin^3 x)+\binom{4n+1}{1}(1-\sin^2 x)^{2n}(\sin x) \quad (6)$$

Noting that

$$\binom{4n+1}{2k}=\binom{4n+1}{(4n+1)-2k}$$, for $k = 0, 1, 2, \ldots\ldots\ldots, 2n$

Then, the expression on the R.H.S of (6) can be rewritten as

Sin (4n+1)x =

$$(\sin^{4n+1}x)-\binom{4n+1}{2}(1-\sin^2 x)(\sin^{4n-1}x)+\ldots.+(-1)^p\binom{4n+1}{2p}(1-\sin^2 x)^{2n-p}(\sin^{2p+1}x)$$

$$...............-\binom{4n+1}{4n-2}(1-\sin^2 x)^{2n-1}(\sin^3 x)+\binom{4n+1}{4n}(1-\sin^2 x)^{2n}(\sin x)$$

$$=\sum_{p=0}^{2n}(-1)^p\binom{4n+1}{2p}(\sin x)^{4n-2p+1}(1-\sin^2 x)^p \quad (7)$$

Thisexpression is the same as expression on the R.H.S of (5) with *cos x* replaced by *sin x.*

we thus have

Sin (4n+1)x = f_{4n+1} *(sin x)*

Hence the result : If *Cos (4n+1) =* f_{4n+1}***(cos x)* then *Sin(4n+1)x =*** f_{4n+1}***(sin x)*** ***(I)***

To establish the result (II)

Again consider

Cos (4n-1)x+ i Sin (4n-1)x = (cos x +i sin x)$^{4n-1}$ (8)

Expandingthe R.H.S of (8) employing Binomial theorem and equating real and imaginary parts on both sides, we get

Cos (4n-1) =

$$\cos^{4n-1}x-\binom{4n-1}{2}(\cos^{4n-3}x)(\sin^2 x)+\binom{4n-1}{4}(\cos^{4n-5}x)(\sin^4 x)$$

$$.........+(-1)^{q-1}\binom{4n-1}{2q}(\cos x)^{4n-2q-1}(\sin^2 x)^q+.........................+\binom{4n-1}{4n-2}(\cos x)(\sin x)^{4n-2}.$$

$$=\sum_{p=0}^{2n}(-1)^p\binom{4n+1}{2q}(\cos x)^{4n-2q-1}(1-\cos^2 x)^q=g_{4n-1}(\cos x)....(say) \quad (9)$$

And

Sin (4n-1)x =

$$\binom{4n-1}{1}(\cos^{4n-2}x)(\sin x)-\binom{4n-1}{3}(\cos^{4n-4}x)(\sin^3 x)+\binom{4n-1}{5}(\cos^{4n-6}x)(\sin^5 x)$$

$$.......+(-1)^q\binom{4n-1}{2q+1}(\cos^{4n-2q-2}x)(\sin^{2q+1}x)...........+\binom{4n-1}{4n-3}(\cos^2 x)(\sin^{4n-3}x)-(\sin^{4n-1}x)$$

By rewriting the R.H.S of the above in the reverse order and noting that

$$\binom{4n-1}{2k}=\binom{4n-1}{(4n-1)-2k}$$, *k = 0,1,2,3,.....................2n-1*

We note that

Sin (4n-1)x=

$$-\quad(\sin^{4n-1}x)-\binom{4n-1}{4n-3}(\cos^2 x)(\sin^{4n-3}x)+......+(-1)^q\binom{4n-1}{2q+1}(\cos^{4n-2q-2}x)(\sin^{2q+1}x)..........$$

$$......-\binom{4n-1}{5}(\cos^{4n-6}x)(\sin^5 x)+\binom{4n-1}{3}(\cos^{4n-4}x)(\sin x)^3-\binom{4n-1}{1}(\cos^{4n-2}x)(\sin x)$$

$$= -\sum_{q=0}^{2n-1} (-1)^q \binom{4n+1}{2q} (\sin x)^{4n-2q-1} (1-\sin^2 x)^q = -g_{4n-1}(\sin x) \tag{10}$$

The expression (10) for *sin (4n-1)x* is same as the expression (9) with *cos x* with replaced by *sin x* but for thea change in sign ..*i.eSin (4n+1)x = -* g_{4n-1}*(sin x).*

Hence the result : If ***Cos (4n-1)x =*** g_{4n-1}***(cos x)*** **then** ***sin (4n-1)x = -*** g_{4n-1}***(sin x)*** ***(II)***

(II) Proof employing the trigonometric properties of complimentary angels.

i) *Sin {(4n+1)x} = Sin {(4n+1)(* $\frac{\pi}{2} - y$ *)}* where x = $\frac{\pi}{2} - y$

= Sin {(2nπ+1)+($\frac{\pi}{2} - (4n+1)y$ *)}*

= Sin { $\frac{\pi}{2}$ *-(4n+1)y}*

= Cos (4n+1)y = f_{4n+1}*(cosy)*

= f_{4n+1}*(cos(*$\frac{\pi}{2} - y$*))*

=f_{4n+1}*(sin x)*

ii) Similarly

Sin {(4n-1)x} = Sin {(4n-1)($\frac{\pi}{2} - y$ *)}*

= Sin {2nπ- $\frac{\pi}{2} - (4n-1)y$ *}*

=Sin { -$\frac{\pi}{2}$ *-(4n-1)y}*

=- Sin { $\frac{\pi}{2}$ *+ (4n-1)y }*

= Cos (4n-1)y

= - g_{4n-1}*(cos y)*

= - g_{4n-1}*(cos*$\left(\frac{\pi}{2} - x\right)$*)*

= - g_{4n-1}*(sin x).*

CONCLUSION

The following results are proved in two ways: Employing De movire's and Binomial theorems and by using the properties of trigonometric ratios of complimentary angles.

i.e. If *Cos (4n+1)x =* f_{4n+1}*(cos x)* then *Sin (4n +1)x =* f_{4n+1} *(sin x)*

Cos (4n-1)x = g_{4n-1}*(cos x) then Sin(4n-1)x = -* g_{4n-1} *(sin x).*

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