International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 1, January - 2014, PP 79-82
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Composite Mapping of Flip-Flop Poset of Join-irreducible Elements of Distributive Lattice

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Abstract: *Every dimensional distributive lattice has a "unique irreducible decomposition" as a join of join irreducible elements. By this spirit here we show that composition mapping of distributive lattice having unique minimal gives the inclusion order of the ideals of sub poset of join irreducible element of distributive lattice. This distributive lattice will be obtained by construction method, as every flip-flop poset of a digraph is distributive lattice. Every join-prime element is also join irreducible, and every meet-prime element is also meet irreducible. The converse holds if L is distributive. In view of this in present paper we deals with subposet of join irreducible element of a distributive lattice and its composite mapping.*

Keywords: *Flip-flop, Vertex Cut, C- orientations, circulation, Join irreducible elements*

1. Introduction

The subject of this paper is join irreducible element of distributive lattice and its composite mapping. Some of the past authors prompt this in their research articles [Felsner, Propp]. Felsner constructs a distributive lattice on those orientations of a planar graph that have fixed out degree on every α-orientation. The α-orientations of a graph generalize f factors, spanning trees, Eulerian orientations and Schnyder woods. Propp presents a method to generate a distributive lattice on those orientations of a graph, which have the number of forward arcs in cycles as c-orientations.

The motivation of the work is based upon the question, whether or with which obstructions composition mapping of distributive lattice having unique minimal gives the inclusion order of the ideals of subposet of join irreducible element of distributive lattice. It turns out that the generalization is possible, but yields a theory which is not as nice as in past. Therefore, we focus on point that every flip-flop poset of a digraph is distributive lattice.

Given an undirected graph [6] G = (V, E) we call a directed graph D = (V, A) an orientation of G if there is a bijection $f : E \rightarrow A$ such that $f(\{u, v\}) \in \{(u, v), (v, u)\}$. If **ω** is an orientation of G then G is unique up to isomorphism and we call the isomorphism class of G denoted by $\underline{\boldsymbol{\omega}}$ of **ω** and for digraphs **ω** , **ω ′** we say that **ω ′** is a reorientation of **ω** if $\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\omega}'}$ Clearly, two digraphs **ω** , **ω ′** are orientations of isomorphic graphs if and only if **ω** and **ω ′** have identical graphs which by definition is equivalent to **ω ′** being a reorientation of **ω**. Propp introduce two operations on directed graphs to obtain a partial order on some of the reorientations of a given digraph **ω** in his paper "lattice structure for orientation of graphs". Fix an arbitrary vertex T of **ω** the forbidden vertex and allow flipping and flopping all the other vertex cuts of **ω**. For reorientations of **ω ′** and **ω″** of **ω**, we define **ω ′** ≤ **ω ″** if **ω ″** can be obtained from **ω ′** by a sequence of flips at vertices different from T. The binary relation ≤ is a partial order on the set of reorientations of **ω** that can be reached from **ω** via a sequence of flips and flops. This set of reorientations of **ω** partially ordered by ≤ is called the flip flop poset $P_{ff}(\vec{D}$, T). A classical result "Every flip-flop poset of a digraph is distributive lattice "shown by Kolja in his paper "Distributive Lattices on Graph Orientations" [6] R. P Dilworth represents a finite distributive lattice D as the congruence lattice of a finite lattice L. While G,Gratzer and E. T Schmidt [5] shown a finite lattice L with O(2^{2n}) elements,where n is number of join irreducible elements of D whose construction technique developed by G,Gratzer and H.Lakser [4]. As by Garrett Birkhoff poset of irreducible generalizes the constructions to provide a representation for finite distributive lattice. Hence Birkhoff's theorem establishes an isomorphism relationship between a given finite distributive lattice L_D and the set of order ideals of a particular subposet $J_D(L_D)$ of join-irreducible elements of the lattice L_D. In general, the poset is obtained from L_D as the set of L_D join-irreducibles, and L_D is recovered from poset up to isomorphism by taking the set of ideals of elements from poset. The significance of this theorem lies in the fact that the poset of join-irreducible of L can be exponentially more succinct than the lattice itself. Since in a distributive lattices the poset of meet-irreducible is isomorphic to the poset of join-irreducible so it is sufficient to work with either the join-irreducible or meet-irreducible.

2. PRELIMINARIES

Throughout the paper L_D will be a distributive lattice & [D] denotes the set of digraphs that can be obtained from D by adding transitive arcs, where an arc (u, v) is called transitive if there is a directed path in D. Let $J_D(L_D)$ denotes the sub poset of join irreducible elements of L_D. If Ø is a mapping from $J_D(L_D)$ to $\frac{V(D)}{\{T\}}$ and φ is a mapping from flipflop poset : $P_{ff}(\vec{D}$, T) to a subset $\vec{Z}\ of\ \frac{V(\vec{D})}{\{T\}}$ then $(\O o\varphi)(P_{ff}(\vec{D}$,T)) gives the inclusion order on the ideals of $J_D(L_D)$. Since V ($\vec{D}$)= V (D) for every $\vec{D} \in$ [D] the bijection Ø can be seen as a bijection from J_D to $\frac{V(\vec{D})}{\{T\}}$. Where T is a terminal vertex.

Join-irreducible Elements

In a lattice, an element x is join-irreducible if x is not the join of a finite set of other elements. Equivalently, x is join-irreducible if it is neither the bottom element of the lattice (the join of zero elements) nor the join of any two smaller elements. In any lattice, a join-prime element must be join-irreducible. Equivalently, an element that is not join-irreducible is not join-prime. For, if an element x is not join-irreducible, there exist smaller y and z such that x = y ∨ z. Then x ≤ y ∨ z, and x is not less than or equal to either y or z, showing that it is not join-prime. There exist lattices in which the join-prime elements form a proper subset of the join-irreducible elements, but in a distributive lattice the two types of elements coincide. For, suppose that x is join-irreducible, and that x ≤ y ∨ z. This inequality is equivalent to the statement that x = x ∧ (y ∨ z), and by the distributive law x = (x ∧ y) ∨ (x ∧ z). However, since x is join-irreducible, at least one of the two terms in this join must be x itself, showing that either x = x ∧ y (equivalently x ≤ y) or x = x ∧ z (equivalently x ≤ z).

Vertex cut

We define a cut of D as an arc set S [x] C A, introduced by X C V. The cut consists of all the arcs that are incident to X & V/X i.e. S [X] = {u, v) ∈ A: {u, v} ∩ X = 1} where maps a set to its cardinality. A cut will be directed if all its arcs point from X either to V/X (positively directed) or from V/X to X (negatively directed). A cut of the form S [{V}] for v ∈ V is called a vertex cut.

Flip-Flop Operation

Reversing the orientation on all the arcs c of a positively directed vertex cut is called a flip. The universe operation i.e. reversing the orientation on a negatively directed vertex cut is called a flop.

Theorem [1.1]: Let $\vec{D} = (V, A)$ be a connected digraph and T∈ V.Then the flip-flop poset $P_{ff}(\vec{D}$, T) with forbidden vertex T is a distributive lattice.

Theorem [1.2]:Birkoff theorem: Any finite distributive lattice L is isomorphic to the lattice of lower sets of the partial order of the join-irreducible elements of L. That is, there is a unique poset P such that L =J (P).

3. THEOREM

Let L_D be a distributive lattice & $\vec{D} \in$ [D]. If $Ø$:$J_D(L_D) \to \frac{V(D)}{\{T\}}$ and $\varphi: P_{ff}(\vec{D}$,T) $\to$ { $\vec{Z}/\vec{Z} \subseteq \frac{V(\vec{D})}{\{T\}}$} then $(Øo\varphi)(P_{ff}(\vec{D}$,T))=$O(J_D(L_D)$

Proof: Since $\vec{D}$ has no negative directed vertex cut apart from T.i.e.no flops can be performed at $\vec{D}$,i.e. $\vec{D}$ is a minimum of $P_{ff}(\vec{D}$,T), by definition of $P_{ff}(\vec{D}$,T).By theorem [1.1] $P_{ff}(\vec{D}$,T) is a distributive lattice, and $P_{ff}(\vec{D}$,T) has a unique minimum, which must be $\vec{D}$.Thus every element of $P_{ff}(\vec{D}$,T) can be reached by a sequence of a flips starting from $\vec{D}$.By definition of mapping φ it maps every $\vec{D'} \in P_{ff}(\vec{D}$,T) to a vertex set $\vec{Z}$ whose vertex cuts {S[V]\V $\in$ $\vec{Z}$ } can be flipped in some order to reach $\vec{D'}$ from $\vec{D}$.Since flipping different vertex sets $\vec{Z}$, $\vec{Z}$' $\subseteq \frac{V(\vec{D})}{\{T\}}$ yields different reoriented arc sets S[$\vec{Z}$],S[$\vec{Z}$'], there is a unique $\vec{Z} \subseteq \frac{V(\vec{D})}{\{T\}}$ whose vertex cuts have to be flipped to obtain $\vec{D'}$,from $\vec{D}$.for every $\vec{Z} \subseteq \frac{V(\vec{D})}{\{T\}}$ whose vertex cuts can be flipped there is $\varphi^{-1}(\vec{Z}) \in P_{ff}(\vec{D}$,T).Every $\vec{Z}$ $\subseteq$V($\vec{D}$) corresponds to an $Z \subseteq J_D(L_D)$ via the bijection Ø given in the construction of [D].Thus Ø maps every set $\vec{Z}$ of vertices whose cuts can be flipped to reach an element of $P_{ff}(\vec{D}$,T) to an order ideal $Z \subseteq J_D(L)$i.e.$Ø(\varphi(P_{ff}(\vec{D}$,T)) $\subseteq O(J_D(L_D) i.e\ (Øo\varphi)(P_{ff}(\vec{D}$,T)) $\subseteq O(J_D(L_D)$. The vertex cuts at the source of $\vec{D}$ are positively directed. However, the sources are connected to T, Which are not allowed to be flipped, so the cuts of the sources can be flipped exactly once. Furthermore, every vertex cut S[v] can be flipped only after the vertex cuts S[u] for u $\prec_{J_D(L_D)}$ have been flipped. Iteratively every vertex cut can be flipped at most as often as the sources. Together we have that every vertex cut can be flipped at most once in a flipping sequence and Ø maps the set of vertices $\vec{Z}$, whose cuts have been flipped in any flip sequence to an ideal $Z\ of\ J_D(L_D)$. Consider an ideal $Z\ of\ J_D(L_D)$ the corresponding vertex set $\vec{Z}$:=Ø$^{-1}(Z)$ induces a directed cut S[$\vec{Z}$] in $\vec{D}$. It is an elementary fact from graph theory that a cycle in a digraph contributes as many forward acres as backward arcs to its intersection with a directed cut. So reversing the orientation on S [$\vec{Z}$] leaves the number of forward arcs among the cycles of $\vec{D}$ invariant. Thus reversing S [$\vec{Z}$] yields a c-reorientation of $\vec{D}$. Every cut in $\vec{D}$ can be flipped at most once. Therefore for every arc (u, v) $\in$ S[$\vec{Z}$] exactly one of the vertex cuts S[u] and S[v] has to be flipped. However, to flip S[v] one must have flipped S[u] as before. This is S[u] and not S[v] must have been flipped. We have u $\in \vec{Z}$ and by construction Ø$^{-1}$ maps the vertices of $\vec{Z}$ incident to S [$\vec{Z}$] to the maxima of Z. So to reverse S[$\vec{Z}$] the vertex cuts in Ø$^{-1}(Z)$ must be flipped i.e. $O(J_D(L_D) \subseteq (Øo\varphi)(P_{ff}(\vec{D}$,T)) Hence $(Øo\varphi)(P_{ff}(\vec{D}$,T))=$O(J_D(L_D)$.

Lemma [3.1]: Let L_D be a distributive lattice & $\vec{D} \in$ [D] then $P_{ff}(\vec{D}$,T) is isomorphic to L_D.

Proof: By using main result of this paper and theorem [1.2] lemma can be proved.

4. CONCLUSION

We have presented an efficient method for inclusion order on the ideals of subposet of join irreducible elements of distributive lattice using Birkoff representation theorem and have extended the work using mapping $Ø\ and\ \varphi$, Ø is a mapping from sub poset of join irreducible element of distributive lattice to vertex cut and φ is a mapping from flipflop poset to a subset of vertex cut ,so that an ideal of sub poset of join irreducible element induces vertex cut on directed cut in set of digraph.

ACKNOWLEDGEMENT

Guidance during preparation of this research paper fills me with a sense of gratitude. Firstly, I express my gratitude to my supervisor Dr. Madhu Tiwari and co supervisor Dr. C. L Parihar for their generous support. The contribution I received from Dr. Sonu Sen and Dr. Parimeeta Chanchani needs to be specially mentioned for submission of this research paper.

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