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Total coloring of $S(n,m)$-graph

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Abstract: *In this paper, we have defined a new graph called $S(n,m)$-graph for evenn $\geq 2m+2$ and for odd $m > 1$ and found the lower and upper bound for the total chromatic number of $S(n,m)$-graphs. We have also found the total chromatic number of$S(n,2)$ for all $n \geq 6$ and $S(n,3)$ for odd $n \geq 7$.*

Keywords: *Total Coloring, $S(n,m)$-graphs*

1. INTRODUCTION

For the past three decades many researchers have worked on total coloring of graphs. Borodin [1] has discussed the total coloring of graphs. Sudha and K.Manikandan [3] have discussed the total coloring and (k,d)-total coloring of prisms Y_n. Prisms Y_n with $2n$ nodes are characterized as generalized Peteresen graphs $P(n,1)$. H.P.Yap [4] also has defined and discussed the total coloring of graphs. We have defined a new graph $S(n,m)$,$n \geq 2m+2$, $m \geq 3$and the definition follows:

The graph $S(n,m)$ consists of n vertices denoted as $v_1, v_2, v_3, \dots, v_n$.The edges are defined as follows:

(i) $v_i$is adjacent to $v_{i+m}$and $v_n$is adjacent to v_1

(ii) $v_i$is adjacent to v_{i+m}if$i+m < n$

(iii) v_i is adjacent to v_{i+n-m} if $i+m \geq n$.

This graph is a quartic graph and it is both Eulerian and Hamiltonian. The concept of this type of a new graph was introduced by S.Sudha.

Definition 1: A total coloring is a coloring on the vertices and edges of a graph such that

(i) no two adjacent vertices have the same color

(ii) no two adjacent edges have the same color

(iii) no edge and its end vertices are assigned with the same color.

In this paper, we have considered the graph $S(n,m)$ and obtained the upper and lower bound for the total chromatic number.

2. TOTAL COLORING OF $S(n,m)$-GRAPHS

Theorem 1: The total-chromatic number $\chi_{tc}(S(n,m))$ is 6 for $n \geq 2m+2$and odd $m \geq 3$.

Proof: Let $v_1, v_2, \dots\dots, v_{\{n-1\}}, v_n$ be the vertices of the graph $S(n,m)$and its edges are defined as

(i) v_i is adjacent to v_{i+m} and v_n is adjacent to v_1

(ii) $v_i$is adjacent to v_{i+m} if $i+m < n$

(iii) v_i is adjacent to v_{i+n-m} if $i+m \geq n$.

Let the coloring set of $S(n, m)$be the set $\{1,2,3, \dots\}$.

We define the function f_1 from $V(S(n, m))$ to the set $\{1,2,3, \dots\}$ as follows:

$$f_1(v_i) = \begin{cases} 1, & i - \text{odd}, 1 \leq i \leq n \\ 2, & i - \text{even}, 1 \leq i \leq n \end{cases}$$

We define the function $f_2$from E(S(n, m)) to the set {1, 2, 3, . . .} as follows:

$$f_2(v_i v_{i+1}) = \begin{cases} 3, & i - \text{odd}, 1 \leq i \leq n-1 \\ 4, & i - \text{even}, 1 \leq i \leq n-1 \end{cases}$$

$$f_2(v_n v_1) = 4$$

$$f_2(v_i v_{i+m}) = \begin{cases} 5, & i - \text{odd}, 1 \leq i \leq n-m \\ 6, & i - \text{even}, 1 \leq i \leq n-m \end{cases}$$

$$f_2(v_i v_{i+n-m}) = \begin{cases} 5, & i - \text{even}, 1 \leq i \leq m \\ 6, & i - \text{odd}, 1 \leq i \leq m \end{cases}$$

By using the above pattern of coloring, the graph S(n,m) admit total coloring. The total-chromatic number for S(n,m), χtc(S(n,m)) = 6.

□

Illustration 1:

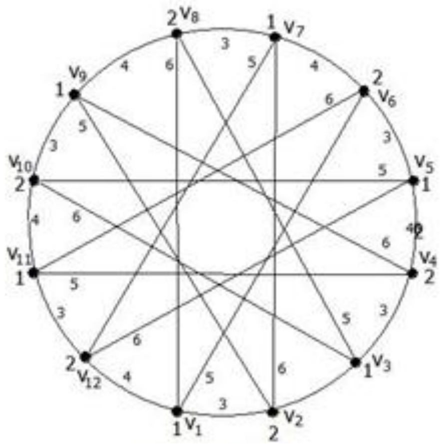

Figure 1. *S(12,5)*

The graph $S(12,5)$ consists of 12 vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}$ which are assigned with the colors 1,2,1,2,1,2,1,2,1,2,1,2 respectively. The outer edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_{12}$ and $v_{12}v_{11}$, $v_{11}v_4, v_4v_9, v_9v_2, v_2v_7, v_7v_{12}, v_{12}v_5, v_5v_{10}, v_{10}v_3,$ are assigned with colors3,4,3,4,3,4,3,4,3,4,3,4 and theinner edges$v_1v_6, v_6v_{11}, v_3v_8, v_3v_8$ are assigned with colors 5,6,5,6,5,6,5,6,5,6,5,6 respectively. The total-chromatic number of$S(12,5)$,$\chi_{tc}(S(12,5)) = 6$.

Theorem 2: The total-chromatic number$\chi_{tc}(S(n, 2))(n \geq 6)$ is 5 for $n \equiv 0 (\text{mod } 6)$ andis 6 for $n \not\equiv 0 (\text{mod } 6)$.

Proof: Let $v_1, v_2, \dots\dots, v_{\{n-1\}}, v_n$be the vertices of the graph $S(n, 2)$ and its edges be denoted by $(v_i v_{i+1}), (v_i v_{i+2}), (v_i v_{i+n-2})$ for $i = 1,2,3, \dots$and $(v_n v_1)$.

Let f_1 be a function that maps $V(S(n, 2))$ to the set $\{1,2,3, \dots\}$ and f_2 be a function that maps $E(S(n, 2))$ to the set $\{1,2,3, \dots\}$ in such a way that f_1 and f_2 satisfy the condition of total coloring.

There are six cases:

(i) $n \equiv 0 (\text{mod } 6)$

(ii) $n \equiv 2(\text{mod } 6)$

(iii) $n \equiv 4(\text{mod } 6)$

(iv)$n \equiv 1(\text{mod } 6)$

(v) $n \equiv 3(\text{mod } 6)$

(vi)$n \equiv 5(\text{mod } 6)$

Case (i): Let $n \equiv 0(\text{mod } 6)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } i \equiv 1(\text{mod } 3), 1 \le i \le n \\ 2, & \text{for all } i \equiv 2(\text{mod } 3), 1 \le i \le n \\ 3, & \text{for all } i \equiv 0(\text{mod } 3), 1 \le i \le n \end{cases}$$

$$f_2(v_i v_{i+1}) = \begin{cases} 4, & i - \text{odd}, 1 \le i \le n-1 \\ 5, & i - \text{even}, 1 \le i \le n-1 \end{cases}$$

$f_2(v_{n-1}v_n) = 2$

$f_2(v_i v_{i+2}) = f_1(v_{i+1}), 1 \le i \le n-2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $S(n,2)$ for $n \equiv 0(\text{mod } 6)$ admit total coloring.

The total-chromatic number of $S(n,2)$, $\chi_{tc}(S(n,2)) = 5$.

Case (ii): Let $n \equiv 2(\text{mod } 6)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } i \equiv 1(\text{mod } 3), 1 \le i \le n-2 \\ 2, & \text{for all } i \equiv 2(\text{mod } 3), 1 \le i \le n-2 \\ 3, & \text{for all } i \equiv 0(\text{mod } 3), 1 \le i \le n-2 \end{cases}$$

$f_1(v_{n-1}) = 4$

$f_1(v_n) = 5$

$$f_2(v_i v_{i+1}) = \begin{cases} 6, & i - \text{odd}, 1 \le i \le n-3 \\ 5, & i - \text{even}, 1 \le i \le n-3 \end{cases}$$

$f_2(v_{n-2}v_{n-1}) = 1$

$f_2(v_{n-1}v_n) = 2$

$f_2(v_n v_1) = 3$

$f_2(v_i v_{i+2}) = f_1(v_{i+1}), 1 \le i \le n-2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $S(n,2)$ admit total coloring.

The total-chromatic number of $S(n,2)$, $\chi_{tc}(S(n,2)) = 6$.

Case(iii): Let $n \equiv 4(\text{mod } 6)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } i \equiv 1(\text{mod } 3), 1 \le i \le n-1 \\ 2, & \text{for all } i \equiv 2(\text{mod } 3), 1 \le i \le n-1 \\ 3, & \text{for all } i \equiv 0(\text{mod } 3), 1 \le i \le n-1 \end{cases}$$

$f_1(v_n) = 4$

$$f_2(v_i v_{i+1}) = \begin{cases} 5, & i - \text{odd}, 1 \le i \le n-1 \\ 6, & i - \text{even}, 1 \le i \le n-1 \end{cases}$$

$f_2(v_n v_1) = 6$

$f_2(v_i v_{i+2}) = f_1(v_{i+1}), 1 \le i \le n-2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $\mathrm{S(n,2)}$ admit total coloring.

The total-chromatic number of $\mathrm{S(n,2)}$, $\chi_{tc}\big(\mathrm{S(n,2)}\big) = 6$.

Case (iv): Let $n \equiv 1(\mathrm{mod}\ 6)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } i \equiv 1(\mathrm{mod}\ 3), 1 \leq i \leq n-1 \\ 2, & \text{for all } i \equiv 2(\mathrm{mod}\ 3), 1 \leq i \leq n-1 \\ 3, & \text{for all } i \equiv 0(\mathrm{mod}\ 3), 1 \leq i \leq n-1 \end{cases}$$

$f_1(v_n) = 4$

$$f_2(v_iv_{i+1}) = \begin{cases} 5, & i - \text{odd}, 3 \leq i \leq n-2 \\ 6, & i - \text{even}, 4 \leq i \leq n-1 \end{cases}$$

$f_2(v_1v_2) = 6$

$f_2(v_2v_3) = 4$

$f_2(v_nv_1) = 5$

$f_2(v_iv_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n-2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $\mathrm{S(n,2)}$ admit total coloring.

The total-chromatic number of $\mathrm{S(n,2)}$, $\chi_{tc}\big(\mathrm{S(n,2)}\big) = 6$.

Case(v): Let $n \equiv 3(\mathrm{mod}\ 6)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } i \equiv 1(\mathrm{mod}\ 3), 1 \leq i \leq n-1 \\ 2, & \text{for all } i \equiv 2(\mathrm{mod}\ 3), 1 \leq i \leq n-1 \\ 3, & \text{for all } i \equiv 0(\mathrm{mod}\ 3), 1 \leq i \leq n-1 \end{cases}$$

$f_1(v_n) = 6$

$$f_2(v_iv_{i+1}) = \begin{cases} 4, & i - \text{odd}, 1 \leq i \leq n-1 \\ 5, & i - \text{even}, 1 \leq i \leq n-1 \end{cases}$$

$f_2(v_nv_1) = 6$

$f_2(v_iv_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n-2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $\mathrm{S(n,2)}$ admit total coloring.

The total-chromatic number of $\mathrm{S(n,2)}$, $\chi_{tc}\big(\mathrm{S(n,2)}\big) = 6$.

Case(vi): Let $n \equiv 5(\mathrm{mod}\ 6)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } i \equiv 1(\mathrm{mod}\ 3), 1 \leq i \leq n-2 \\ 2, & \text{for all } i \equiv 2(\mathrm{mod}\ 3), 1 \leq i \leq n-2 \\ 3, & \text{for all } i \equiv 0(\mathrm{mod}\ 3), 1 \leq i \leq n-2 \end{cases}$$

$f_1(v_{n-1}) = 4$

$f_1(v_n) = 5$

$$f_2(v_iv_{i+1}) = \begin{cases} 6, & i - \text{odd}, 1 \leq i \leq n-2 \\ 5, & i - \text{even}, 1 \leq i \leq n-2 \end{cases}$$

$f_2(v_{n-1}v_n) = 2$

$f_2(v_nv_1) = 3$

$f_2(v_iv_{i+2}) = f_1(v_{i+1}), 1 \leq i \leq n-2$

$f_2(v_{n-1}v_1) = f_1(v_n)$

By using the above pattern, the graph $S(n,2)$ admit total coloring.

The total-chromatic number of $S(n,2)$, $\chi_{tc}(S(n,2)) = 6$.

Illustration2:

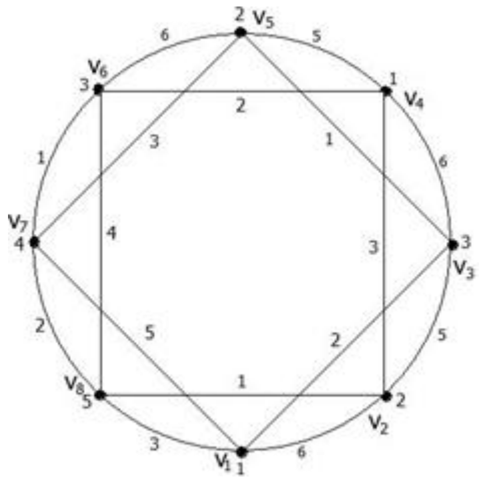

Figure 2. *S(8.2)*

The graph $S(8,2)$ consists of 8 vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ which are assigned with the colors 1,2,3,1,2,3,4,5 respectively. The outer edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8$ and v_8v_1 are assigned with colors 6,5,6,5,6,1,2,3 and the inner edges $v_1v_3, v_3v_5, v_5v_7, v_7v_1, v_2v_4, v_4v_6, v_6v_8, v_8v_2$ are assigned with colors 2,1,3,5,3,2,4,1 respectively. The total-chromatic number of $S(8,2)$, $\chi_{tc}(S(8,2)) = 6$.

Theorem 3: The total-chromatic number $(\chi_{tc}(S(n,3))$ is 7 for $n \equiv 0 (\mathrm{mod}\ 6)$ and is 6 for $n \not\equiv 0 (\mathrm{mod}\ 6)$.

Proof: Let $v_1, v_2, \ldots\ldots, v_{n-1}, v_n$ be the vertices of the graph $S(n,3)$ and its edges be denoted by$(v_i v_{i+1}), (v_i v_{i+2}), (v_i v_{i+n-2})$ for $i = 1,2,3,\ldots$and $(v_n v_1)$.

Let f_1 be a function that maps $V(S(n,3))$ to the set $\{1,2,3,\ldots\}$ and f_2 be a function that maps $E(S(n,3))$ to the set $\{1,2,3,\ldots\}$ in such a way that f_1 and f_2 satisfy the condition of total coloring.

Case(i): For odd $n \geq 7$ and $n \not\equiv 0(\mathrm{mod}\ 3)$

$$f_1(v_i) = \begin{cases} 1, & \text{for all } 1 \leq i \leq n-4, i-\text{odd} \\ 2, & \text{for all } 1 \leq i \leq n-3, i-\text{even} \\ 3, & \text{for all } i = n-2, n \end{cases}$$

$$f_1(v_{n-1}) = 4$$

$$f_2(v_i v_{i+1}) = \begin{cases} 3, \text{for all } 1 \leq i \leq n-4, i-\text{odd} \\ 4, \text{for all } 1 \leq i \leq n-3, i-\text{even} \end{cases}$$

$$f_2(v_{n-2}v_{n-1}) = 2$$

$$f_2(v_{n-1}v_n) = 1$$

$$f_2(v_n v_1) = 2$$

The edges of the form $f_2(v_i v_{i+3})$ for $i = 1,2,3,\ldots,(n-3)$. Takes the coloring pattern as 5,6,5,6, … ,5,6.

The last three of the edges are colored given below.

$$f_2(v_{n-8}v_{n-5}) = 5$$

$$f_2(v_{n-5}v_{n-2}) = 1$$

$$f_2(v_{n-2}v_1) = 6$$

By using the above pattern, the graph $S(n,3)$admit total coloring.

The total-chromatic number of $S(n,3)$, $\chi_{tc}\big(S(n,3)\big) = 6$.

Case(ii): For odd $n \geq 9$ andn $\equiv 0(\text{mod } 3)$

$$f_1(v_i) = \begin{cases} 4, \text{for all } 1 \leq i \leq n-4, i-\text{odd} \\ 5, \text{for all } 1 \leq i \leq n-3, i-\text{even} \end{cases}$$

$f_1(v_{n-2}) = 3$

$f_1(v_{n-1}) = 4$

$f_1(v_n) = 3$

$$f_2(v_iv_{i+1}) = \begin{cases} 3, \text{for all } 1 \leq i \leq n-4, i-\text{odd}, 1 \leq i \leq n-3 \\ 5, \text{for all } 1 \leq i \leq n-3, i-\text{even}, 1 \leq i \leq n-3 \end{cases}$$

$f_2(v_{n-2}v_{n-1}) = 2$

$f_2(v_{n-1}v_n) = 1$

$f_2(v_nv_1) = 2$

$$f_2(v_iv_{i+3}) = \begin{cases} 5, \quad i-\text{odd}, 1 \leq i \leq n-3 \\ 6, \quad i-\text{even}, 1 \leq i \leq n-3 \end{cases}$$

$f_2(v_iv_{i+n-3}) = 7, 1 \leq i \leq 3.$

Now with this type of coloring, the graph $S(n,3)$is total coloring.

The total-chromatic number of $S(n,3)$, $\chi_{tc}\big(S(n,3)\big) = 6$.

Illustration 3:

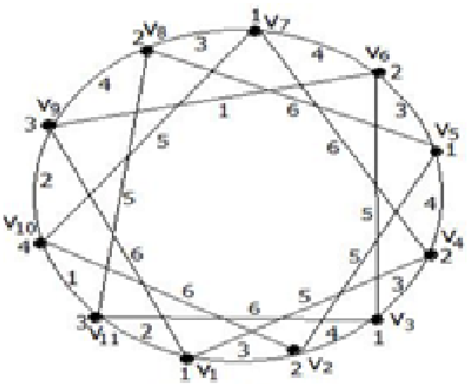

Figure3. *S(11,3)*

The graph S(11,3) consists of 11 vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$ which are assigned with the colors 1,2,1,2,1,2,1,2,3,4,3 respectively. The outer edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_{11}$ and $v_{11}v_1$ are assigned with colors 3,4,3,4,3, 4,3,4,2,1,2 and the inner edges $v_1v_4, v_4v_7, v_7v_{10}, v_{10}v_2, v_2v_5, v_5v_8, v_8v_{11}, v_{11}v_3, v_3v_6, v_6v_9, v_9v_1$ are assigned with colors 5,6,5,6,5, 6,5,6,5,1,6respectively. The total-chromatic number of S(11,3), $\chi_{tc}(\square(11,3)) = 6$.

3. Conclusion

We have found that the lower and upper bound for the total chromatic number of $S(n,m)$, in general, satisfies $5 \leq \chi_{tc}(S(n,m)) \leq 7$. The total-chromatic number for $S(n,m)$ when m takes the value 2 and 3 are also discussed.

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