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Some Properties of Induced Fuzzy and Induced Anti Fuzzy Subgroups on a HX Group

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Abstract: *In this paper, we define the concept of induced fuzzy subgroup (fuzzy HX subgroup) and induced anti fuzzy subgroup (anti fuzzy HX subgroup) of a HX group. We also define a fuzzy coset of a fuzzy HX subgroup, anti fuzzy HX subgroup of a HX group, conjugate fuzzy HX subgroups, conjugate anti fuzzy HX subgroups of a HX group and discussed some of its properties with the examples. Then we establish the relation between the fuzzy subset of a group and fuzzy HX subgroup, and the relation between a fuzzy subset of a group and an anti fuzzy HX subgroup of a HX group. Further we define the level subsets and lower level subsets of a fuzzy cosets of a fuzzy HX subgroup, anti fuzzy HX subgroup and discussed some of its properties. We also define quotient HX subgroup of a HX group induced by a fuzzy normal HX subgroup and by an anti fuzzy normal HX subgroup of a HX group. We also discussed its properties under isomorphism and anti isomorphism.*

Keywords: *HX group , Fuzzy HX subgroup, anti fuzzy HX subgroup, fuzzy cosets of a fuzzy HX group, fuzzy middle cosets, conjugate fuzzy HX subgroups , and quotient HX subgroup*

AMS Subject Classification (2000): 20N25, 03E72, 03F055, 06F35, 03G25.

1. Introduction

The notion of fuzzy sets was introduced by L.A. Zadeh [18]. Fuzzy set theory has been developed in many directions by many researchers and has evoked great interest among mathematicians working in different fields of mathematics. In 1971, Rosenfield [14] introduced the concept of fuzzy subgroup. R. Biswas [1] introduced the concept of anti fuzzy subgroup of group. Li Hongxing [2] introduced the concept of HX group and Chengzhong et al. [4] introduced the concept of fuzzy HX group. Palaniappan.N. et al. [12] discussed the concepts of Anti fuzzy group and its Lower level subgroups. Muthuraj. R., et al. [8] discussed the concepts of Anti Q-fuzzy HX group and its lower level subgroups. Bhattacharya [11] introduced fuzzy right coset and fuzzy left coset of a group. Malik et al.[6] discussed the concepts of fuzzy cosets and quotient semigroups. W. B. Vasantha kandasamy [16] introduced the concept of fuzzy middle cosets of a group. In this paper we define the concept of fuzzy cosets of a fuzzy HX subgroup and anti fuzzy HX subgroup of a HX group, fuzzy middle cosets of a fuzzy HX subgroup and anti fuzzy HX subgroup of a HX group and discussed some of their related properties.

2. Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G, *) is a finite group, e is the identity element of G, and xy, we mean x * y.

2.1 Definition [18]

Let X be any non empty set. A fuzzy subset μ of X is a function μ: X → [0, 1].

2.2 Definition [14]

A fuzzy set μ on G is called fuzzy subgroup of G if for x, y $\in$G,

i. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
ii. $\mu(x^{-1}) = \mu(x)$.

2.3 Definition [1]

A fuzzy set μ on G is called an anti fuzzy subgroup of G if for x, y $\in$G,

i. $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$
ii. $\mu(x^{-1}) = \mu(x)$.

Note: [1, Proposition 3.2]

A fuzzy set μ on G is fuzzy subgroup of G iff μ^c is an anti fuzzy subgroup of G.

2.4 Definition [6]

Let μ be a fuzzy subset of a group G. For any a $\in$ G, aμ defined by $(a\mu)(x) = \mu(a^{-1}x)$ for every x $\in$ G is called the fuzzy coset of G determined by a and μ.

Remark:

i. If a= e, then fuzzy coset aμ = μ
ii. If μ is a fuzzy subgroup of a group G, and a = e then fuzzy coset (aμ) is also a fuzzy subgroup of G.
iii. If μ is an anti fuzzy subgroup of a group G, and a = e then fuzzy coset (aμ) is also an anti fuzzy subgroup of G.

2.1 Example

Let G = {1, i, -1, -i} be a group with respect to multiplication. Define the fuzzy subset μ on G as μ (1) = 0.8, μ (-1) = 0.6, μ (i) = 0.4 the and μ (-i) = 0.4. Clearly μ is a fuzzy subgroup of G. Then the fuzzy coset (-1) μ is defined as

$$(-1)\mu\ (x) = \begin{cases} 0.6 \ \ if\ \ x = 1 \\ 0.8 \ \ if\ \ x = -1 \\ 0.4\ if\ \ x = i \\ 0.4\ if\ \ x = -i \end{cases}$$

Similarly, we can define the other cosets also. The fuzzy coset aμ is a fuzzy subgroup of G if a = e.

We can give an example for fuzzy coset of an anti fuzzy subgroup of a group G.

2.5 Definition [16]

Let μ be a fuzzy subgroup (an anti fuzzy subgroup) of a group G. Then for any a, b $\in$ G, a fuzzy middle coset aμb of a fuzzy subgroup (an anti fuzzy subgroup) μ determined by the elements a and b of G is defined as $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ for every x $\in$ G.

2.2 Example

G = {1, i, -1, -i} be a group with respect to multiplication. Define the fuzzy subset μ on G as μ(1) = 0.8, μ(-1) = 0.6, μ(i) = 0.4 and μ(-i) = 0.4. Clearly μ is a fuzzy subgroup of G.The fuzzy middle coset (-iμi) is defined as

$$(-i\mu\ i)(x) = \begin{cases} 0.8 \ \ if\ \ x = 1 \\ 0.6 \ \ if\ \ x = -1 \\ 0.4\ if\ \ x = i \\ 0.4\ if\ \ x = -i \end{cases}$$

Clearly, (-iμi) is a fuzzy subgroup of G, since $b = i = a^{-1}$.

But

$$(1\mu\, i)(x) = \begin{cases} 0.4 \;\; if \;\; x = 1 \\ 0.4 \;\; if \;\; x = -1 \\ 0.8 \; if \;\; x = i \\ 0.6 \; if \;\; x = -i \end{cases}$$

The fuzzy middle coset (1μi) is not a fuzzy subgroup of G.

Remark:

i. Let μ be the fuzzy subgroup of a group G, then fuzzy middle coset (aμb) is also a fuzzy subgroup of G if $b = a^{-1}$.
ii. Let μ be an anti fuzzy subgroup of a group G, then fuzzy middle coset (aμb) is also an anti fuzzy subgroup of G if $b = a^{-1}$.

2.6 Definition [16]

Let λ and μ be two fuzzy subgroups (anti fuzzy subgroups) of a group G. Then λ and μ are said to be conjugate fuzzy subgroups (conjugate anti fuzzy subgroups) of G if for some $y \in G$, $\lambda(x) = \mu(y^{-1}xy)$ for every $x \in G$.

2.7 Definition [7]

Let G be a group. A fuzzy subgroup (an anti fuzzy subgroup) μ of G is called normal if $\mu(x) = \mu(y^{-1}\, x\, y)$ for all $x, y \in G$.

2.8 Definition [16]

Let μ be a fuzzy subgroup of a group G and $H = \{x \in G / \mu(x) = \mu(e)\}$ then o(μ), order of μ is defined as o(μ) = o(H).

2.9 Definition

Let $(G_1, +)$ and $(G_2, *)$ be any two groups. A map $f: G_1 \rightarrow G_2$ is said to be an anti homomorphism if $f(a + b) = f(b) * f(a)$ for all $a, b \in G_1$.

2.10 Definition

Let $(G_1, +)$ and $(G_2, *)$ be any two groups. $f: G_1 \rightarrow G_2$ is said to be an anti isomorphism if

i. f is one-one (that is, if f (x) = f (y) then x = y)
ii. f is onto (that is, range of f = co domain of f)
iii. $f(a + b) = f(b) * f(a)$ for all $a, b \in G_1$.

3. Induced Fuzzy Subgroups (Fuzzy HX Subgroups) on a HX Group and their Properties

In this section, we define induced fuzzy subgroup and induced anti fuzzy subgroup of a HX group and discussed some of their related results. Also we define the concepts of fuzzy cosets, fuzzy middle cosets of a fuzzy HX subgroup and anti fuzzy HX subgroup of a HX group. We also define conjugate fuzzy HX subgroup, conjugate anti fuzzy HX subgroup of a HX group and discussed some of their related properties.

3.1 Definition [2]

Let G be a finite group. In $2^G - \{\phi\}$, a nonempty set $\vartheta \subset 2^G - \{\phi\}$ is called a HX group on G, if ϑ is a group with respect to the algebraic operation defined by $AB = \{ab / a \in A \text{ and } b \in B\}$, which its unit element is denoted by E.

3.2 Definition

Let μ be a fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. A fuzzy set λ^{μ} defined on ϑ is said to be a fuzzy subgroup induced by μ on ϑ or a fuzzy HX subgroup on ϑ, if , for any $A, B \in \vartheta$,

i. $\lambda^{\mu}(AB) \geq \min\{\lambda^{\mu}(A), \lambda^{\mu}(B)\}$

ii. $\lambda^{\mu}(A^{-1}) = \lambda^{\mu}(A)$.

where, $\lambda^{\mu}(A) = \max\{\mu(x) / \text{for all } x \in A \subseteq G\}$.

3.1 Theorem

If μ is a fuzzy subgroup of G then the fuzzy subset λ^{μ} is a fuzzy HX subgroup on ϑ.

Proof

Let μ be a fuzzy subgroup of G.

$$\begin{aligned}
\min\{\lambda^{\mu}(A), \lambda^{\mu}(B)\} &= \min\{\max\{\mu(x) / \text{for all } x\in A \subseteq G\}, \max\{\mu(y)/ \text{ for all } y\in B\subseteq G\}\} \\
&= \min\{\mu(x_0), \mu(y_0)\} \\
&\leq \{\mu(x_0y_0)\}, \text{ since } \mu \text{ is a fuzzy subgroup of G.} \\
&\leq \{\max\{\mu(xy) / \text{for all } xy\in AB \subseteq G\} \\
&\leq \lambda^{\mu}(AB)
\end{aligned}$$

Hence, $\lambda^{\mu}(AB) \geq \min\{\lambda^{\mu}(A), \lambda^{\mu}(B)\}$.

$$\begin{aligned}
\lambda^{\mu}(A) &= \max\{\mu(x) / \text{ for all } x \in A \subseteq G\} \\
&= \max\{\mu(x^{-1}) / \text{ for all } x \in A \subseteq G\} \\
&= \max\{\mu(x^{-1}) / \text{ for all } x^{-1} \in A^{-1} \subseteq G\} \\
&= \lambda^{\mu}(A^{-1})
\end{aligned}$$

$\lambda^{\mu}(A) = \lambda^{\mu}(A^{-1})$.

Hence, λ^{μ} is a fuzzy HX subgroup of ϑ.

3.1 Example

Let G = { 1 , -1, i , -i } be a group under the binary operation multiplication and define a fuzzy set μ on G as $\mu(1) = 0.8$, $\mu(-1) = 0.6$, $\mu(i) = 0.4$ and $\mu(-i) = 0.4$. Let $\vartheta = \{\{1,-1\}, \{i, -i\}\}$.

Clearly $(\vartheta, \cdot)$ is a HX group.

Define $\lambda^{\mu}(A) = \max\{\mu(x) / \text{ for all } x \in A \subseteq G\}$.

Clearly $\lambda^{\mu}(\{1,-1\}) = 0.8$ and $\lambda^{\mu}(\{i,-i\}) = 0.4$ and if for all A , B $\in \vartheta$,

i. $\lambda^{\mu}(AB) \geq \min\{\lambda^{\mu}(A), \lambda^{\mu}(B)\}$

ii. $\lambda^{\mu}(A^{-1}) = \lambda^{\mu}(A)$.

Clearly, λ^{μ} is a fuzzy HX subgroup of ϑ.

Remark:

Let μ be a fuzzy subset of a group G. If λ^{μ} is a fuzzy HX subgroup on ϑ, then μ need not be a fuzzy subgroup of G.

3.2 Example:

Let G = { 1 , -1, i , -i } be a group under the binary operation multiplication and fuzzy subset μ on G defined as $\mu(1) = 0.8$, $\mu(-1) = 0.6$, $\mu(i) = 0.4$ and $\mu(-i) = 0.25$. Let $\vartheta = \{\{1,-1\},\{i,-i\}\}$ be a HX group. Define a fuzzy subset λ^{μ} on ϑ such that $\lambda^{\mu}(A) = \max\{\mu(x)/ \text{ for all } x \in A \subseteq G\}$. Then $\lambda^{\mu}(\{1,-1\}) = 0.8$, $\lambda^{\mu}(\{i, -i\}) = 0.4$. Clearly λ^{μ} is a fuzzy HX subgroup of ϑ, but μ is not a fuzzy subgroup of G.

3.3 Definition

Let μ be a fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. A fuzzy set λ_{μ} defined on ϑ is said to be an anti fuzzy subgroup induced by μ on ϑ or an anti fuzzy HX subgroup on ϑ, if , for any A, B $\in \vartheta$,

i. $\lambda_{\mu}(AB) \leq \max\{\lambda_{\mu}(A), \lambda_{\mu}(B)\}$

ii. $\lambda_{\mu}(A^{-1}) = \lambda_{\mu}(A)$.

where, $\lambda_{\mu}(A) = \min\{\mu(x) / \text{for all } x \in A \subseteq G\}$.

3.2 Theorem

If μ is an anti fuzzy subgroup of G then λ_{μ} is also an anti fuzzy HX subgroup on ϑ.

Proof

Let μ be an anti fuzzy subgroup of G. By note 2.1, μ^c is a fuzzy subgroup of G. By theorem 3.1 λ^{μ^c} is a fuzzy HX subgroup on ϑ. Clearly $(\lambda^{\mu^c})^c$ is an anti fuzzy HX subgroup on ϑ. Now $(\lambda^{\mu^c})^c$ (x) = 1 - λ^{μ^c} (x) = λ_μ (x). Hence λ_μ is an anti fuzzy HX subgroup on ϑ.

3.3 Example

Let G = { 1 , -1, i , -i } be a group under the binary operation multiplication and define a fuzzy set μ on G as $\mu(1) = 0.4$, $\mu(-1) = 0.6$, $\mu(i) = 0.7$ and $\mu(-i) = 0.7$. Let $\vartheta = \{\{1,-1\}, \{i, -i\}\}$.

Clearly $(\vartheta, \cdot)$ is a HX group.

Define $\lambda_\mu(A)= \min \{ \mu(x) /$ for all $x \in A \subseteq G\}$.

Clearly, $\lambda_\mu(\{1,-1\})= 0.4$ and $\lambda_\mu(\{i,-i\})= 0.7$ and for all A , B $\in \vartheta$,

i. $\lambda_\mu(AB) \leq \max \{ \lambda_\mu(A), \lambda_\mu(B) \}$
ii. $\lambda_\mu(A^{-1}) = \lambda_\mu(A)$.

Clearly, λ_μ is an anti fuzzy HX subgroup of ϑ.

Remark:

If μ is a fuzzy subset of a group G and λ_μ be an anti fuzzy HX subgroup defined on ϑ, a HX group on G such that $\lambda_\mu(A) = \min \{ \mu(x) /$ for all $x \in A \subseteq G\}$, then μ need not be an anti fuzzy subgroup of G.

3.4 Example:

Let G = { 1 , -1, i , -i } be a group under the binary operation multiplication and fuzzy set μ on G defined as $\mu(1) = 0.25$, $\mu(-1) = 0.6$, $\mu(i) = 0.75$ and $\mu(-i) = 0.7$. Let $\vartheta = \{\{1,-1\}, \{i,-i\}\}$ be a HX group. Define a fuzzy subset λ_μ on ϑ such that $\lambda_\mu(A)= \min\{\mu(x)/$for all $x\in A \subseteq G\}$ which gives $\lambda_\mu(\{1,-1\}) = 0.25$, $\lambda_\mu(\{i, -i\}) = 0.7$. Clearly λ_μ is an anti fuzzy HX subgroup of ϑ, but μ is not an anti fuzzy subgroup of G.

3.3 Theorem

Let μ be a fuzzy set defined on G and let λ^μ be a fuzzy HX subgroup of a HX group ϑ, then the fuzzy coset $(A\lambda^\mu)$ is a fuzzy HX subgroup of ϑ if $\min\{\lambda^\mu(A^{-1}Y), \lambda^\mu(A)\} = \lambda^\mu(A^{-1}Y)$ for every $A \in \vartheta$.

Proof

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ.

For every X and Y in ϑ, we have,

$$\begin{aligned}
(A\lambda^\mu)(XY^{-1}) &= \lambda^\mu(A^{-1}XY^{-1}) \\
&= \lambda^\mu(A^{-1}XY^{-1}AA^{-1}) \\
&\geq \min\{\lambda^\mu(A^{-1}X), \lambda^\mu(Y^{-1}AA^{-1})\} \\
&\geq \min\{\lambda^\mu(A^{-1}X), \min\{\lambda^\mu(Y^{-1}A), \lambda^\mu(A^{-1})\}\} \\
&= \min\{\lambda^\mu(A^{-1}X), \min\{\lambda^\mu((Y^{-1}A)^{-1}), \lambda^\mu(A)\}\} \\
&= \min\{\lambda^\mu(A^{-1}X), \min\{\lambda^\mu(A^{-1}Y), \lambda^\mu(A)\}\} \\
&= \min\{\lambda^\mu(A^{-1}X), \lambda^\mu(A^{-1}Y)\} \\
&= \min\{(A\lambda^\mu)(X), (A\lambda^\mu)(Y)\}
\end{aligned}$$

Therefore, $(A\lambda^\mu)(XY^{-1}) \geq \min\{(A\lambda^\mu)(X), (A\lambda^\mu)(Y)\}$

Hence, $(A\lambda^\mu)$ is a fuzzy HX subgroup of ϑ.

3.4 Theorem

Let μ be a fuzzy set defined on G and let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ, then the fuzzy coset $(A\lambda_\mu)$ is also an anti fuzzy HX subgroup of ϑ if $\max\{ \lambda_\mu(A^{-1}Y), \lambda_\mu(A)\}= \lambda_\mu(A^{-1}Y)$ for every $A \in \vartheta$.

Proof

Let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ.

For every X and Y in ϑ, we have,

$$
\begin{aligned}
(A\lambda_\mu)(XY^{-1}) &= \lambda_\mu(A^{-1}XY^{-1}) \\
&= \lambda_\mu(A^{-1}XY^{-1}AA^{-1}) \\
&\leq \max\{\lambda_\mu(A^{-1}X), \lambda_\mu(Y^{-1}AA^{-1})\} \\
&\leq \max\{\lambda_\mu(A^{-1}X), \max\{\lambda_\mu(Y^{-1}A), \lambda_\mu(A^{-1})\} \\
&= \max\{\lambda_\mu(A^{-1}X), \max\{\lambda_\mu((Y^{-1}A)^{-1}), \lambda_\mu(A)\}\} \\
&= \max\{\lambda_\mu(A^{-1}X), \max\{\lambda_\mu(A^{-1}Y), \lambda_\mu(A)\}\} \\
&= \max\{\lambda_\mu(A^{-1}X), \lambda_\mu(A^{-1}Y)\} \\
&= \max\{(A\lambda_\mu)(X), (A\lambda_\mu)(Y)\}\}
\end{aligned}
$$

Therefore, $(A\lambda_\mu)(XY^{-1}) \leq \{\max\{(A\lambda_\mu)(X), (A\lambda_\mu)(Y)\}\}$

Hence $(A\lambda_\mu)$ is an anti fuzzy HX subgroup of ϑ.

3.4 Definition

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ. For any A , B $\in\vartheta$, define a binary relation ***R*** on ϑ by ,

$$A \, \boldsymbol{R} \, B \iff \lambda^\mu(AB^{-1}) = \lambda^\mu(E), \text{ E is the identity element of } \vartheta.$$

3.5 Definition

Let λ^μ be a fuzzy normal HX subgroup of ϑ.Then the equivalence class ***R*** containing A is denoted by $\lambda^\mu{}_A$ and ϑ/λ^μ denotes the corresponding quotient set, defined as $\vartheta/\lambda^\mu = \{\lambda^\mu{}_A \ / \ A \in \vartheta\}$.

3.6 Definition

Let λ^μ be a fuzzy normal HX subgroup of a HX group (ϑ,∗) , then the quotient set ϑ/λ^μ is a quotient HX group induced by λ^μ with the operation $\lambda^\mu_A * \lambda^\mu_B = \lambda^\mu_{A*B} = \lambda^\mu_{AB}$.

3.5 Example

Let G = { 1 , -1, i , -i } be a group under the binary operation multiplication and define a fuzzy subset μ on G as $\mu(1) = 0.8$, $\mu(-1) = 0.6$, $\mu(i) = 0.4$ and $\mu(-i) = 0.4$. Let $\vartheta = \{\{1\},\{-1\},\{i\},\{-i\}\}$ be a HX group. Define a fuzzy subset λ^μ on ϑ as $\lambda^\mu(A)= \max\{\mu(x) \ / \text{ for all } x \in A \subseteq G\}$ which gives $\lambda^\mu(\{1\}) = 0.8$, $\lambda^\mu(\{-1\}) = 0.6$, $\lambda^\mu(\{i\}) = 0.4$ and $\lambda^\mu(\{-i\}) = 0.4$. Clearly λ^μ is a fuzzy normal HX subgroup of ϑ. Define a fuzzy quotient set ϑ/λ^μ as $\vartheta/\lambda^\mu = \{\lambda^\mu_{\{1\}}, \lambda^\mu_{\{-1\}}\}$. Clearly ϑ/λ^μ is a quotient HX group.

3.7 Definition

Let λ_μ be an anti fuzzy normal HX subgroup of ϑ. Then the equivalence class ***R*** containing A is denoted by λ_{μ_A} and $\vartheta/\lambda_{\mu_A}$ denotes the corresponding quotient set, defined as $\vartheta/\lambda_\mu = \{\lambda_{\mu_A} \ / \ A \in\vartheta\}$.

3.8 Definition

Let λ_μ be an anti fuzzy normal HX subgroup of a HX group (ϑ,∗) , then the quotient set $\vartheta/\lambda_{\mu_A}$ is a quotient HX group induced by λ_μ with the operation $\lambda_{\mu_A} * \lambda_{\mu_B} = \lambda_{\mu_{A*B}} = \lambda_{\mu_{AB}}$.

3.6 Example

Let G = { 1 , -1, i , -i } be a group under the binary operation multiplication and define a fuzzy set μ on G as $\mu(1) = 0.2$, $\mu(-1) = 0.4$, $\mu(i) = 0.6$ and $\mu(-i) = 0.6$. Let $\vartheta = \{\{1\},\{-1\},\{i\},\{-i\}\}$ be HX group. Define a fuzzy subset λ_μ on ϑ such that $\lambda_\mu(A)= \min\{\mu(x) \ / \text{ for all } x \in A \subseteq G\}$ which gives $\lambda_\mu(\{1\}) = 0.2$, $\lambda_\mu(\{-1\}) = 0.4$, $\lambda_\mu(\{i\}) = 0.6$ and $\lambda_\mu(\{-i\}) = 0.6$. Clearly λ_μ is an anti fuzzy normal HX subgroup of ϑ. Define a fuzzy quotient set ϑ/λ_μ as $\vartheta/\lambda_\mu = \{\lambda_{\mu_{\{1\}}}, \lambda_{\mu_{\{-1\}}}\}$. Clearly ϑ/λ_μ is a quotient HX group.

3.9 Definition

Let μ be a fuzzy subset defined on G and let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ with $\lambda_\mu(A) = \min\{\mu(x) / \text{for all } x \in A \subseteq G\}$. For any $A \in \vartheta$, $A\lambda_\mu$ defined by $A\lambda_\mu(X) = \lambda_\mu(A^{-1}X)$ for every $X \in \vartheta$ is called the fuzzy coset of a fuzzy HX group λ_μ of ϑ determined by A.

3.7 Example

Let G be the Klein's 4 group. Then G = {e, a, b, ab} where $a^2 = e = b^2$, $ab = ba$ and e the identity element of G.

Define a fuzzy subset μ on G as follows,

$$\mu(x) = \begin{cases} \frac{1}{2} & if\ \ x = e \\ \frac{3}{4} & if\ \ x = a \\ \frac{1}{4} & if\ \ x = b,\ ab \end{cases}$$

Let $\vartheta = \{\{e,a\}, \{b, ab\}\}$ be a HX group on G.

Define as $\lambda^\mu(A) = \max\{\mu(x) / \text{for all } x \in A \subseteq G\}$.

We have,

$$\lambda^\mu(X) = \begin{cases} \frac{3}{4} & if\ \ X = \{e, a\} \\ \frac{1}{4} & if\ \ X = \{b, ab\} \end{cases}$$

Clearly, λ^μ is a fuzzy HX subgroup of a HX group ϑ. Now, we compute the fuzzy cosets of λ^μ.

i. $$\{e,a\}\lambda^\mu(X) = \begin{cases} \frac{3}{4} & if\ \ X = \{e, a\} \\ \frac{1}{4} & if\ \ X = \{b, ab\} \end{cases}$$

ii. $$\{b,ab\}\lambda^\mu(X) = \begin{cases} \frac{1}{4} & if\ \ X = \{e, a\} \\ \frac{3}{4} & if\ \ X = \{b, ab\} \end{cases}$$

Remark:

i. If A = E, then fuzzy coset $A\lambda^\mu = \lambda^\mu$.
ii. If A = E, then fuzzy coset $A\lambda_\mu = \lambda_\mu$.
iii. If λ^μ is a fuzzy HX subgroup of a group ϑ, and If A = E, then fuzzy coset $A\lambda^\mu$ is also a fuzzy HX subgroup of ϑ.
iv. If λ_μ is an anti fuzzy HX subgroup of a group ϑ, and If A = E, then fuzzy coset $A\lambda_\mu$ is also an anti fuzzy HX subgroup of ϑ.

3.10 Definition

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ. Then for any $A, B \in \vartheta$, a fuzzy middle coset $(A\lambda^\mu B)$ of a fuzzy HX subgroup λ^μ of ϑ determined by A and B is defined as $(A\lambda^\mu B)(X) = \lambda^\mu(A^{-1}X B^{-1})$ for every $X \in \vartheta$.

Similarly we can define middle coset for an anti fuzzy HX subgroup λ_μ.

3.5 Theorem

Let μ be a fuzzy subset defined on G and let λ^μ be a fuzzy HX subgroup of a HX group ϑ, then the fuzzy middle coset $(A\lambda^\mu A^{-1})$of a fuzzy HX group λ^μ of ϑ determined by A and A^{-1} of ϑ is a fuzzy HX subgroup of ϑ.

Proof

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ and $A \in \vartheta$.

Let $X, Y \in \vartheta$. Then,

$(A\lambda^{\mu}A^{-1})\,(XY^{-1}) = \lambda^{\mu}\,(A^{-1}XY^{-1}A)$, by the definition

$= \lambda^{\mu}\,(A^{-1}X\,A\,A^{-1}Y^{-1}A)$

$= \lambda^{\mu}\,((A^{-1}XA)\,(A^{-1}YA)^{-1})$

$\geq \min\{\lambda^{\mu}(A^{-1}XA), \lambda^{\mu}(A^{-1}YA)^{-1}\}$

$\geq \min\{\lambda^{\mu}(A^{-1}X\,A), \lambda^{\mu}(A^{-1}YA)\}$,

{since λ^{μ} is a fuzzy HX subgroup of ϑ}

$\geq \min\{(A\lambda^{\mu}A^{-1})(X), (A\lambda^{\mu}A^{-1})(Y)\}$.

Therefore, $(A\lambda^{\mu}A^{-1})\,(XY^{-1}) \geq \min\{(A\lambda^{\mu}A^{-1})(X), (A\lambda^{\mu}A^{-1})(Y)\}$.

Hence, $(A\lambda^{\mu}A^{-1})$ is an anti fuzzy HX subgroup of a HX group ϑ.

3.6 Theorem

Let μ be a fuzzy subset defined on G and let λ_{μ} be an anti fuzzy HX subgroup of a HX group ϑ, then the fuzzy middle coset $(A\lambda_{\mu}A^{-1})$ of an anti fuzzy HX group λ_{μ} of ϑ determined by A and A^{-1} of ϑ is an anti fuzzy HX subgroup of ϑ.

Proof

Let λ_{μ} be an anti fuzzy HX subgroup of a HX group ϑ and $A \in \vartheta$.

Let $X, Y \in \vartheta$. Then,

$(A\lambda_{\mu}A^{-1})\,(XY^{-1}) = \lambda_{\mu}\,(A^{-1}XY^{-1}A)$, by the definition

$= \lambda_{\mu}\,(A^{-1}X\,A\,A^{-1}Y^{-1}A)$

$= \lambda_{\mu}\,((A^{-1}XA)\,(A^{-1}YA)^{-1})$

$\leq \max\{\lambda_{\mu}(A^{-1}XA), \lambda_{\mu}(A^{-1}YA)^{-1}\}$

$\leq \max\{\lambda_{\mu}(A^{-1}X\,A), \lambda_{\mu}(A^{-1}YA)\}$,

{since λ_{μ} is an anti fuzzy HX subgroup of ϑ}

$\leq \max\{(A\lambda_{\mu}A^{-1})(X), (A\lambda_{\mu}A^{-1})(Y)\}$.

Therefore, $(A\lambda_{\mu}A^{-1})\,(XY^{-1}) \leq \max\{(A\lambda_{\mu}A^{-1})(X), (A\lambda_{\mu}A^{-1})(Y)\}$.

Hence, $(A\lambda_{\mu}A^{-1})$ is an anti fuzzy HX subgroup of a HX group ϑ.

3.11 Definition

Let μ and η be the two fuzzy sets defined on G. Let λ^{μ} and γ^{η} be any two fuzzy HX subgroups of a HX group ϑ. Then λ^{μ} and γ^{η} are said to be conjugate fuzzy HX subgroups of ϑ if for some $A\in\vartheta$, $\lambda^{\mu}(X) = \gamma^{\eta}(A^{-1}XA)$ for every $X \in \vartheta$.

Similarly we can define the same for anti fuzzy HX subgroups of ϑ.

3.12 Definition

Let ϑ be a HX group. A fuzzy HX subgroup λ^{μ} of ϑ is said to be normal if for all

$A, B \in \vartheta$, $\lambda^{\mu}(ABA^{-1}) = \lambda^{\mu}(B)$ or $\lambda^{\mu}(AB) = \lambda^{\mu}(BA)$.

3.7 Theorem

λ^{μ} and γ^{η} be conjugate fuzzy HX subgroups of an abelian HX group ϑ if and only if $\lambda^{\mu} = \gamma^{\eta}$.

Proof

Let λ^{μ} and γ^{η} be any two fuzzy HX subgroups of an abelian HX group ϑ, then for some $B \in \vartheta$, we have,

$\lambda^{\mu}(A) = \gamma^{\eta}(B^{-1}AB)$, for every $A \in \vartheta$

$= \gamma^{\eta}(B^{-1}BA)$, since ϑ is an abelian HX group

$= \gamma^{\eta}(EA) = \gamma^{\eta}(A)$.

Therefore, $\lambda^{\mu}(A) = \gamma^{\eta}(A)$ for every $A \in \vartheta$. Hence $\lambda = \gamma$.

Conversely, let $\lambda^{\mu} = \gamma^{\eta}$, then ,

we have, $\lambda^{\mu}(A) = \gamma^{\eta}(A) = \gamma^{\eta}(E^{-1}EA) = \gamma^{\eta}(E^{-1}AE)$, for every $A \in \vartheta$.

Hence, λ^{μ} and γ^{η} are conjugate fuzzy HX subgroups of ϑ.

3.8 Theorem

λ_μ and γ_η be conjugate anti fuzzy HX subgroups of an abelian HX group ϑ if and only if $\lambda_\mu = \gamma_\eta$.

Proof

It is clear from theorem 3.7.

3.13 Definition [8]

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ. For any $t \in [0,1]$, the set $U(\lambda^\mu; t) = \{ A \in \vartheta / \lambda^\mu(A) \geq t \}$ is called the level subset of λ^μ.

3.14 Definition

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ. For any $t \in [0,1]$, the set $U(A\lambda^\mu ; t) = \{ X \in \vartheta / (A\lambda^\mu)(X) = \lambda^\mu(A^{-1}X) \geq t$, for some $A \in \vartheta\}$ is called the level subset of a fuzzy coset $A\lambda^\mu$.

3.15 Definition [8]

Let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ. For any $t \in [0,1]$, the set $L(\lambda_\mu ; t) = \{ A \in \vartheta / \lambda_\mu(A) \leq t \}$ is called the lower level subset of λ_μ.

3.16 Definition

Let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ. For any $t \in [0,1]$, the set $L(A\lambda_\mu ; t) = \{ X \in \vartheta / (A\lambda_\mu)(X) = \lambda_\mu(A^{-1}X) \leq t$, for some $A \in \vartheta \}$ is called the lower level subset of a fuzzy coset $A\lambda_\mu$.

3.9 Theorem

Let μ be a fuzzy subset defined on G and let λ^μ be a fuzzy HX subgroup of a HX group ϑ, then $U(A\lambda^\mu ; t) = AU(\lambda^\mu ; t)$, for every $A \in \vartheta$ and $t \in [0, 1]$.

Proof

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ and let $X \in \vartheta$.

$$\begin{aligned} \text{Now, } X \in U(A\lambda^\mu ; t) &\Leftrightarrow (A\lambda^\mu)(X) \geq t \\ &\Leftrightarrow \lambda^\mu(A^{-1}X) \geq t \\ &\Leftrightarrow A^{-1}X \in U(\lambda^\mu ; t) \\ &\Leftrightarrow X \in AU(\lambda^\mu ; t) \end{aligned}$$

Hence, $U(A\lambda^\mu ; t) = AU(\lambda^\mu ; t)$, for every $A \in \vartheta$.

3.10 Theorem

Let μ be a fuzzy set defined on G and let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ, then $L(A\lambda_\mu ; t) = AL(\lambda_\mu ; t)$, for every $A \in \vartheta$ and $t \in [0, 1]$.

Proof

It is clear from theorem 3.9.

3.11 Theorem

Let μ be a fuzzy set defined on G and let λ^μ be a fuzzy HX subgroup of a HX group ϑ. Then $X\lambda^\mu = Y\lambda^\mu$, for X and Y in ϑ if and only if $\lambda^\mu(X^{-1}Y) = \lambda^\mu(Y^{-1}X) = \lambda^\mu(E)$.

Proof

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ.

Let $X\lambda^\mu = Y\lambda^\mu$, for X and Y in ϑ.

Then, $X\lambda^\mu(X) = Y\lambda^\mu(X)$ and $X\lambda^\mu(Y) = Y\lambda^\mu(Y)$, which implies that,

$\lambda^\mu(X^{-1} X) = \lambda^\mu(Y^{-1} X)$ and $\lambda^\mu(X^{-1} Y) = \lambda^\mu(Y^{-1} Y)$.

Hence $\lambda^\mu(E) = \lambda^\mu(Y^{-1} X)$ and $\lambda^\mu(X^{-1} Y) = \lambda^\mu(E)$

Therefore, $\lambda^{\mu}(Y^{-1} X) = \lambda^{\mu}(X^{-1} Y) = \lambda^{\mu}(E)$.

Conversely, let $\lambda^{\mu}(Y^{-1} X) = \lambda^{\mu}(X^{-1} Y) = \lambda^{\mu}(E)$, for X and Y in ϑ.

For every A in ϑ , we have,

$$\begin{aligned} X\lambda^{\mu}(A) &= \lambda^{\mu}(X^{-1}A) \\ &= \lambda^{\mu}(X^{-1} Y Y^{-1}A) \\ &\geq \min\{\lambda^{\mu}(X^{-1} Y), \lambda^{\mu}(Y^{-1}A)\} \\ &= \min\{\lambda^{\mu}(E), \lambda^{\mu}(Y^{-1}A)\} \\ &= \lambda^{\mu}(Y^{-1}A) \\ &= Y\lambda^{\mu}(A). \end{aligned}$$

Therefore, $X\lambda^{\mu}(A) \geq Y\lambda^{\mu}(A)$.

similarly, $Y\lambda^{\mu}(A) \geq X\lambda^{\mu}(A)$.

Hence,$X\lambda^{\mu}(A) = Y\lambda^{\mu}(A)$.

Hence, $X\lambda^{\mu} = Y\lambda^{\mu}$.

3.12 Theorem

Let μ be a fuzzy set defined on G and let λ_{μ} be an anti fuzzy HX subgroup of a HX group ϑ. Then $X\lambda_{\mu} = Y\lambda_{\mu}$, for X and Y in ϑ if and only if $\lambda_{\mu}(X^{-1}Y) = \lambda_{\mu}(Y^{-1}X) = \lambda_{\mu}(E)$.

Proof

Let λ_{μ} be a fuzzy HX subgroup of a HX group ϑ.

Let $X\lambda_{\mu} = Y\lambda_{\mu}$, for X and Y in ϑ .

Then, $X\lambda_{\mu}(X) = Y\lambda_{\mu}(X)$ and $X\lambda_{\mu}(Y) = Y\lambda_{\mu}(Y)$,which implies that,

$\lambda_{\mu}(X^{-1} X) = \lambda_{\mu}(Y^{-1} X)$ and $\lambda_{\mu}(X^{-1} Y) = \lambda_{\mu}(Y^{-1} Y)$.
Hence $\lambda_{\mu}(E) = \lambda_{\mu}(Y^{-1} X)$ and $\lambda_{\mu}(X^{-1} Y) = \lambda_{\mu}(E)$

Therefore, $\lambda_{\mu}(Y^{-1} X) = \lambda_{\mu}(X^{-1} Y) = \lambda_{\mu}(E)$.

Conversely, let $\lambda_{\mu}(Y^{-1} X) = \lambda_{\mu}(X^{-1} Y) = \lambda_{\mu}(E)$, for X and Y in ϑ.

For every A in ϑ , we have,

$$\begin{aligned} X\lambda_{\mu}(A) &= \lambda_{\mu}(X^{-1}A) \\ &= \lambda_{\mu}(X^{-1} Y Y^{-1}A) \\ &\leq \max\{\lambda_{\mu}(X^{-1} Y), \lambda_{\mu}(Y^{-1}A)\} \\ &= \max\{\lambda_{\mu}(E), \lambda_{\mu}(Y^{-1}A)\} \\ &= \lambda_{\mu}(Y^{-1}A) \\ &= Y\lambda_{\mu}(A). \end{aligned}$$

Therefore, $X\lambda_{\mu}(A) \leq Y\lambda_{\mu}(A)$.

similarly, $Y\lambda_{\mu}(A) \leq X\lambda_{\mu}(A)$.

Hence, $X\lambda_{\mu}(A) = Y\lambda_{\mu}(A)$.

Hence, $X\lambda_{\mu} = Y\lambda_{\mu}$.

3.13 Theorem

Let μ be a fuzzy set defined on G and let λ_{μ} be an anti fuzzy HX subgroup of a HX group ϑ and $X\lambda_{\mu} = Y\lambda_{\mu}$, for X and Y in ϑ. Then $\lambda_{\mu}(X) = \lambda_{\mu}(Y)$.

Proof

Let λ_{μ} be an anti fuzzy HX subgroup of a HX group ϑ and $X\lambda_{\mu} = Y\lambda_{\mu}$, for X and Y in ϑ.

$$\begin{aligned} \text{Now, } \lambda_{\mu}(X) &= \lambda_{\mu}(YY^{-1}X) \\ &\leq \max\{\lambda_{\mu}(Y), \lambda_{\mu}(Y^{-1}X)\} \\ &= \max\{\lambda_{\mu}(Y), \lambda_{\mu}(E)\}, \text{ by Theorem 3.11} \\ &= \lambda_{\mu}(Y). \end{aligned}$$

Therefore, $\lambda_\mu(X) \leq \lambda_\mu(Y)$.

Similarly, $\lambda_\mu(Y) \leq \lambda_\mu(X)$.

Hence, $\lambda_\mu(X) = \lambda_\mu(Y)$.

3.14 Theorem

Let μ be a fuzzy set defined on G and let λ^μ be a fuzzy HX subgroup of a HX group ϑ and $X\lambda^\mu = Y\lambda^\mu$, for X and Y in ϑ. Then $\lambda^\mu(X) = \lambda^\mu(Y)$.

Proof

It is clear from theorem 3.13.

3.15 Theorem

Let μ be a fuzzy set defined on G and let λ^μ be a fuzzy HX subgroup of a HX group ϑ and $U(X\lambda^\mu ; t) = U(Y\lambda^\mu ; t)$, for $X, Y \in \vartheta - U(\lambda^\mu ; t)$ and $t \in [0, 1]$, then $\lambda^\mu(X) = \lambda^\mu(Y)$.

Proof

Let λ^μ be a fuzzy HX subgroup of a HX group ϑ and $U(X\lambda^\mu ; t) = U(Y\lambda^\mu ; t)$, for $X, Y \in \vartheta - U(\lambda^\mu ; t)$, and $t \in [0, 1]$.

But $Y^{-1}X$ and $X^{-1}Y \in U(\lambda^\mu ; t)$). (by theorem 3.12)

$$\begin{aligned} \text{Now, } \lambda^\mu(X) &= \lambda^\mu(YY^{-1}X) \\ &\geq \min\{\lambda^\mu(Y), \lambda^\mu(Y^{-1}X)\} \\ &= \min\{\lambda^\mu(Y), t\}, \\ &= \lambda^\mu(Y). \end{aligned}$$

Therefore, $\lambda^\mu(X) \geq \lambda^\mu(Y)$.

Similarly, $\lambda^\mu(Y) \geq \lambda^\mu(X)$.

Hence, $\lambda^\mu(X) = \lambda^\mu(Y)$.

3.16 Theorem

Let μ be a fuzzy set defined on G and let λ_μ be an anti fuzzy HX subgroup of a HX group ϑ and $L(X\lambda_\mu ; t) = L(Y\lambda_\mu ; t)$, for $X, Y \in \vartheta - L(\lambda_\mu ; t)$ and $t \in [0, 1]$, then $\lambda_\mu(X) = \lambda_\mu(Y)$.

Proof

It is clear from theorem 3.15.

3.17 Theorem

Let μ be a fuzzy set defined on G and let λ^μ be a fuzzy normal HX subgroup of a HX group $(\vartheta, *)$, then the quotient set ϑ / λ^μ is a HX group with the operation $\lambda_A^\mu * \lambda_B^\mu = \lambda_{A*B}^\mu = \lambda_{AB}^\mu$.

Proof

Let λ_A^μ and $\lambda_B^\mu \in \vartheta / \lambda^\mu$ for some A and B $\in \vartheta$,

Clearly, $B^{-1} \in \vartheta$.

Therefore, $\lambda_{B^{-1}}^\mu \in \vartheta / \lambda^\mu$.

Now, $\lambda_A^\mu * \lambda_{B^{-1}}^\mu = \lambda_{AB^{-1}}^\mu \in \vartheta / \lambda^\mu$ as $AB^{-1} \in \vartheta$.

Hence ϑ / λ^μ is a HX group.

3.18 Theorem

Let μ be a fuzzy set defined on G and let λ_μ be a anti fuzzy normal HX subgroup of a HX group $(\vartheta, *)$, then the quotient set $\vartheta / \lambda_{\mu_A}$ is a HX group with the operation $\lambda_{\mu_A} * \lambda_{\mu_B} = \lambda_{\mu_{A*B}} = \lambda_{\mu_{AB}}$.

Proof

It is clear from theorem 3.17.

3.19 Theorem

If f: $\vartheta \to \vartheta'$ be a homomorphism of HX group ϑ onto a HX group ϑ' and let λ_μ, γ_η be two anti fuzzy normal HX subgroups of ϑ and ϑ' with $\lambda_\mu(A) = f^{-1}(\gamma_\eta)(A) = \gamma_\eta(f(A))$. Then the function g: $\vartheta / \lambda_\mu \to \vartheta' / \gamma_\eta$ defined as g(λ_{μ_A}) = $\gamma_{\eta_{f(A)}}$, for every $A \in \vartheta$, is an isomorphism on a HX group ϑ / λ_μ .

Proof

Let λ_{μ_A} and λ_{μ_B} $\in \vartheta / \lambda_\mu$.

Now , g(λ_{μ_A}) = g(λ_{μ_B})

$$\Leftrightarrow \gamma_{\eta_{f(A)}} = \gamma_{\eta_{f(B)}}$$

$$\Leftrightarrow \gamma_\eta(f(A)(f(B))^{-1}) = \gamma_\eta(E'),\ E' \text{ is the identity of } \vartheta'$$

$$\Leftrightarrow \gamma_\eta(f(A)f(B^{-1}) = \gamma_\eta(E'),\ f \text{ is a homomorphism}$$

$$\Leftrightarrow \gamma_\eta(f(AB^{-1}) = \gamma_\eta(f(E)),\ f \text{ is a homomorphism}$$

$$\Leftrightarrow \lambda_\mu(AB^{-1}) = \lambda_\mu(E)$$

$$\Leftrightarrow \lambda_{\mu_A} = \lambda_{\mu_B}$$

Hence g is one-one .

Clearly g is onto as f is onto.

Now ,

$$\begin{aligned} g(\lambda_{\mu_A}.\lambda_{\mu_B}) &= g(\lambda_{\mu_{AB}}) \\ &= \gamma_{\eta_{f(AB)}} \\ &= \gamma_{\eta_{f(A)f(B)}},\ f \text{ is a homomorphism} \\ &= \gamma_{\eta_{f(A)}}.\gamma_{\eta_{f(B)}} \\ &= g(\lambda_{\mu_A}).g(\lambda_{\mu_B}) \end{aligned}$$

$$\text{Therefore } g(\lambda_{\mu_A}.\lambda_{\mu_B}) = g(\lambda_{\mu_A}).g(\lambda_{\mu_B})$$

Hence g is an isomorphism.

3.20 Theorem

If f: $\vartheta \to \vartheta'$ be a homomorphism of HX group ϑ onto a HX group ϑ' and let λ^μ , γ^η be two fuzzy normal HX subgroups of ϑ and ϑ' with $\lambda^\mu(A) = f^{-1}(\gamma^\eta)(A) = \gamma^\eta(f(A))$. Then the function g: $\vartheta / \lambda^\mu \to \vartheta' / \gamma^\eta$ defined as g(λ_A^μ) = $\gamma_{f(A)}^\eta$, for every $A \in \vartheta$, is an isomorphism on a HX group ϑ / λ^μ .

Proof

It is clear from theorem 3.19.

3.21 Theorem

If f: $\vartheta \to \vartheta'$ be an anti homomorphism of HX group ϑ onto a HX group ϑ' and λ^μ , γ^η be two fuzzy normal HX subgroups of ϑ and ϑ' with $\lambda^\mu(A) = f^{-1}[\gamma^\eta(A)] = \gamma^\eta(f(A))$. Then the function g: $\vartheta / \lambda^\mu \to \vartheta' / \gamma^\eta$ defined as g(λ_A^μ) = $\gamma_{f(A)}^\eta$, for every $A \in \vartheta$, is an anti isomorphism on a HX group ϑ / λ .

Proof

Let λ_A^μ and $\lambda_B^\mu \in \vartheta / \lambda^\mu$.

Now, g(λ_A^μ) = g(λ_B^μ)

$$\Leftrightarrow \gamma^\eta_{f(A)} = \gamma^\eta_{f(B)}$$

$$\Leftrightarrow \gamma^\eta(f(A)(f(B))^{-1}) = \gamma^\eta(E'),\ E' \text{ is the identity of } \vartheta'$$

$$\Leftrightarrow \gamma^\eta(f(A)f(B^{-1}) = \gamma^\eta(E'),\ f \text{ is a homomorphism}$$

$$\Leftrightarrow \gamma^\eta(f(AB^{-1}) = \gamma^\eta(f(E))\ f \text{ is a homomorphism}$$

$$\Leftrightarrow \lambda^\mu(AB^{-1}) = \lambda^\mu(E)$$

$$\Leftrightarrow \lambda_A^\mu = \lambda_B^\mu$$

Hence g is one-one .

Clearly g is onto as f is onto.

Now,

$$
\begin{aligned}
g(\lambda_A^\mu.\lambda_A^\mu) &= g(\lambda_{AB}^\mu) \\
&= \lambda^\eta_{f(AB)} \\
&= \gamma^\eta_{f(A)f(B)},\ f \text{ is a homomorphism} \\
&= \lambda^\eta_{f(A)}.\lambda^\eta_{f(B)} \\
&= g(\lambda_A^\mu).g(\lambda_B^\mu)
\end{aligned}
$$

$$\text{Therefore } g(\lambda_A^\mu.\lambda_B^\mu) = g(\lambda_A^\mu).g(\lambda_B^\mu)$$

Hence g is an anti isomorphism.

3.22 Theorem

If f: $\vartheta \to \vartheta'$ be an anti homomorphism of HX group ϑ onto a HX group ϑ' and λ_μ, γ_η be two anti fuzzy normal HX subgroups of ϑ and ϑ' with $\lambda_\mu(A) = f^{-1}(\gamma_\eta)(A) = \gamma_\eta(f(A))$. Then the function g: $\vartheta / \lambda_\mu \to \vartheta' / \gamma_\eta$ defined as g(λ_{μ_A}) = $\gamma_{\eta_{f(A)}}$, for every $A \in \vartheta$, is an anti isomorphism on a HX group ϑ / λ_μ .

Proof

It is clear from theorem 3.21.

4. Conclusion

1. Here we redefined the fuzzy HX subgroup and discussed some of their related properties with the suitable examples, which will be very useful in the application field of fuzzy groups.
2. We discussed the relationship between fuzzy and anti fuzzy HX subgroups of a HX group, it guide us to learn to apply fuzzy HX subgroups in the various fields of engineering and technology.
3. Also we discussed the concepts of pseudo fuzzy cosets of an induced fuzzy and induced anti fuzzy subgroups of a HX subgroup, which provide the expanded idea of fuzzy HX subgroups for the well travel in the field of communication.

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