International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 1, Issue 3, December - 2013, PP 178-190
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Some Elementary Problems from the Note Books of Srinivasa Ramanujan - Part (I)

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The life of Srinivasa Ramanujan and his Mathematics have frequently been enshrouded in an aura of mystery. He occupies a central but significant position in the History of Mathematics.

Sriman Srinivasa Ramanujan Ayengar (1887-1920) is a self taught mathematical prodigy. As his mentor Prof. Hardy rightly puts it – he is a natural genius. Despite little formal training, just at school level he produced many and enigmatic results that baffle seasoned mathematicians of all times. He exhibits in his contributions, his flair for recognition of wonderful patterns in Numbers – Magic Squares – Solutions /Roots of Polynomial Equations and Simultaneous Equations Algebraic Identities etc., in his typical idiosyncratic manner even before he went on searching for Diverging Series – Series Inversions – Continued Fractions – Elliptic Integrals – Infinite Integrals – Approximations to π and many more such advanced topics in Mathematics Modular Functions – Equations like Class – Invariants, Theta Functions etc., beyond anybody's intuition and comprehension. Several of S.R's results are classical and well known (re – discoveries most likely the origins/references unknown to him) but S.R offers many interesting new results that show S.R at his best. His development of the subject is unique and characteristic of his own without a trace of influence by any other author. He does not even use the standard notations at several places. It appears, he did not bother much about the Mathematical regour in his earlier presentations might be because of his eagerness I conceiving the end results. At some stances, especially in dealing with Divergent series, he jotted down a few fallacious results.

Despite all this, S.R. discovered the basic underlying formulas for several theories, a few of which were developed earlier and some evolved later. It is remarkable that Ramanujans formulas are almost invariably correct, even though his methods, in general, are without a sound theoretical foundation/Mathematical regour. Serious Mathematicians like G.H.Hardy, J.E.Littlewood, Watson, Wilson, Bruce C Berndt, R.A.Askey and several others were unable to find the roads that led S.R to his discoveries. All these he could envisage during the short span of his life of just thirty two (+) years.

During the period 1903 – 1914, he was scribbling / jotting down his results, invariably without providing any proof, in three small note books which are popularly referred as **Manuscripts of Srinivasa Ramanujan**. These we refer in this presentation as **S.R.Ms. (parts 1, 2, 3)**. It is reported that the proof for each problem was written on a slate by S.R, wiping out the proof after making an entry in a note book. Hints for quite a few problems, placed at random can also be seen here and there. Entries in these books are highly disorganized and not properly graded. S.R. scribed his results in these books (S.R.Ms.).before he left for England in 1914. The facsimiles of these fascinating invaluable notebooks were published in two volumes by T.I.F.R during the memorable centenary year 1957 of Madras University and made them accessible to the world at large. These volumes are popularly known as **Note Books of Srinivasa Ramanujan (NBSR).Vol.I** contains S.R.Ms. part 1 and Vol. II contains S.R.Ms. part 2 and part 3. These were reprinted recently in 2012, marking the 125th birth anniversary of Srinivasa Ramanujan. Several years after the demise of SR in 1920 a bunch of papers containing the contribution of SR during the last years of his life were located by Prof. G. Andrews at Trinity college library and these papers constitute **Lost Note Book** of SR.

On a careful survey of these volumes, one gets amazed in finding some exciting entries, sprinkled here and there, disorderly placed, and that can be discussed even at the school / under graduate level. It is an eye-brow-raising / eye-catching experience to notice such simple problems (beyond the comprehension of a routine Mathematics Teacher) scribed astray amongst problems that can be established only with great struggle in wilderness.

It would be a rich fruitful exercise to the Mathematics Teachers if they can put in some effort to bring down some of the simple problems of S.R to Schools – base level institutions – Schools, Under graduate and Postgraduate institutions. That would inspire their trainees in understanding the spirit of Ramanujan. This would incidentally remove a prevailing misnomer that all the contributions of S.R are beyond the scope and reach of school / college teachers and students.

A scour in this direction is very much wanted to popularize at least some of the S.R's problems in our teaching institutions ranging from Foundation to Post Graduation levels. It is high time that Mathematics Associations / Clubs in the country to conduct more and more Seminars and Workshops to popularize S.R. problems in the Teaching Community at large. Efforts be made by the syllabus makers for inclusion in the curriculum that would encourage Text Book writers to discuss in their works some properly graded select problems of S.R. appending simple exercises if not extensions.

As on today S.R. is remembered invariably on 22nd December every year (his birth date), by recapping just some anecdotes from his life history and paying homage to the departed soul with scant (technical) reference to any of his problems elementary or advanced ones. The purpose of this presentation is to place before the readers some elementary problems from the S.R Note Books that can be popularized in Teaching Institutions at different levels in the country and world wide at large.

§§ Problem 1. SRMs (2) p.305 and NBSR Vol II p 348

$$\frac{1}{x+a_1}+\frac{a_1}{(x+a_1)(x+a_2)}+\frac{a_1a_2}{(x+a_1)(x+a_2)(x+a_3)} \quad \textbf{and so on up to n terms}$$

$$=\frac{1}{x}-\frac{1}{x\left(1+\frac{x}{a_1}\right)\left(1+\frac{x}{a_2}\right)\left(1+\frac{x}{a_3}\right)\ldots\ldots\ldots\left(1+\frac{x}{a_n}\right)} \qquad *$$

Proof:

The first term on the L.H.S of $* = \frac{1}{x+a_1}=\frac{1}{x}-\left\{\frac{1}{x}-\frac{1}{x+a_1}\right\}=\frac{1}{x}-\frac{a_1}{x(x+a_1)}$

Sum of the first two terms on the L.H.S of *

$$=\left\{\frac{1}{x}-\frac{a_1}{x(x+a_1)}\right\}+\frac{a_1}{(x+a_1)(x+a_2)}$$

$$=\frac{1}{x}-a_1\left\{\frac{1}{x(x+a_1)}-\frac{1}{(x+a_1)(x+a_2)}\right\} = \frac{1}{x}-\frac{a_1a_2}{x(x+a_1)(x+a_2)}$$

Sum of the first three terms on the L.H.S. of *

$$=\left\{\frac{1}{x}-\frac{a_1a_2}{x(x+a_1)(x+a_2)}\right\}+\frac{a_1a_2}{x(x+a_1)(x+a_2)(x+a_3)}$$

$$=\frac{1}{x}-a_1a_2\left\{\frac{1}{x(x+a_1)(x+a_2)}-\frac{1}{x(x+a_1)(x+a_2)(x+a_3)}\right\}$$

$$=\frac{1}{x}-\frac{a_1a_2a_3}{x(x+a_1)(x+a_2)(x+a_3)}$$

Similarly the sum of first four terms on the L.H.S. of *

$$=\frac{1}{x}-\frac{a_1a_2a_3a_4}{x(x+a_1)(x+a_2)(x+a_3)(x+a_4)} \text{ and so on.}$$

We thus notice that the L.H.S of * = Sum of the n terms on the L.H.S. of *.

$$=\frac{1}{x}-\frac{a_1a_2a_3\ldots\ldots\ldots\ldots a_{n-1}a_n}{x(x+a_1)(x+a_2)(x+a_3)\ldots\ldots\ldots\ldots(x+a_{n-1})(x+a_n)}$$

$$=\frac{1}{x}-\frac{1}{x\left(1+\frac{x}{a_1}\right)\left(1+\frac{x}{a_2}\right)\left(1+\frac{x}{a_3}\right)\ldots\ldots\ldots\left(1+\frac{x}{a_n}\right)} \quad = \text{R.H.S. of } *$$

This establishes the result *.

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Entries in SRMs (III) p 15 involving e and π

$e^{\pi\sqrt{22}} = 2508951.9982$

$e^{\pi\sqrt{37}} = 199148647.999978....$

$e^{\pi\sqrt{58}} = 24591257751.99999982.....$

$e^{\pi\sqrt{163}} = 262537412640768746.99999999999925$

Q.J.Pure & Appl. Maths Vol.45 (1913 – 1914)

§§ Problem 2: Ms (1)p4 & (2)p7 and NBSR Vol I p 7 & also Vol II p 13

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots\ldots\ldots+\frac{1}{2n}$$
$$=\frac{n}{2n+1}+\frac{1}{2^3-2}+\frac{1}{4^3-4}+\frac{1}{6^3-6}+\ldots\ldots+\frac{1}{(2n)^3-2n} \quad *$$

and deduce that $\log_e 2=\frac{1}{2}+\frac{1}{2^3-2}+\frac{1}{4^3-4}+\frac{1}{6^3-6}+\ \ldots\ldots$ **and so on.**

Solution:

S. R uses the identity: $\frac{1}{x^3-x}=\frac{1}{2(x-1)}-\frac{1}{x}+\frac{1}{2(x+1)}$ ---------------- (2.1)

in establishing the reset result * and also some more for the coming results.

$$\text{R.H.S. of } * = \frac{n}{n+1}+\sum_{k=1}^{n}\frac{1}{(2k)^3-(2k)} \qquad (\because x = 2k)$$

$$=\frac{n}{2n+1}+\sum_{k=1}^{n}\frac{1}{(2k-1)(2k)(2k+1)} \qquad \text{---------------- (2.2)}$$

$$=\frac{n}{2n+1}+\sum_{k=1}^{n}\left\{\frac{1}{2(2k-1)}-\frac{1}{2k}+\frac{1}{2(2k+1)}\right\} \qquad \text{using (2.1)}$$

$$=\frac{n}{2n+1}+\frac{1}{2}\left\{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ \ldots\ldots+\frac{1}{2n-1}\right\}-\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ \ldots\ldots+\frac{1}{2n}\right\}$$
$$+\frac{1}{2}\left\{\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ \ldots\ldots+\frac{1}{2n+1}\right\}$$

$$=\frac{n}{2n+1}+\frac{1}{2}\left\{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ \ldots\ldots+\frac{1}{2n-1}\right\}-\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ \ldots\ldots+\frac{1}{2n}\right\}$$

$$=\frac{n}{2n+1}+\frac{1}{2}\left\{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ \ldots\ldots+\frac{1}{2n-1}\right\}-\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ \ldots\ldots+\frac{1}{2n}\right\}$$
$$+\frac{1}{2}\left\{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ \ldots\ldots+\frac{1}{2n-1}\right\}-\frac{1}{2}+\frac{1}{2n+1}$$

$$=\left\{\frac{n}{2n+1}-\frac{1}{2}+\frac{1}{2n+1}\right\}+\left\{\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ \ldots\ldots+\frac{1}{2n-1}\right\}-\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ \ldots\ldots+\frac{1}{2n}\right\}$$

$$=\{0\}+\left\{\begin{array}{l}\left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\ldots\ldots\ldots+\frac{1}{2n-1}+\frac{1}{2n}\right]\\ -\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\ldots+\frac{1}{2n}\right]-\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\ldots+\frac{1}{2n}\right]\end{array}\right\}$$

$$=\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ \ldots+\frac{1}{n}\right\}+\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ \ldots+\frac{1}{2n}\right\}-2\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ \ldots\ldots+\frac{1}{2n}\right\}$$

$$=\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right\}+\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right\}-\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\ \ldots+\frac{1}{n}\right\}$$

$$=\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ \ldots\ldots+\frac{1}{2n}\right\}=L.H.S\ of\ * \qquad \text{------- (2.3)}$$

The result * is thus established.

Deduction for $\log_e 2$: We thus have

$$\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ \dots\dots+\frac{1}{2n}\right\}=\frac{n}{2n+1}+\left\{\frac{1}{2^3-2}+\frac{1}{4^3-4}+\frac{1}{6^3-6}+\ \dots\dots+\frac{1}{(2n)^3-2n}\right\}$$

$$i.e.,\sum_{k=1}^{n}\frac{1}{n+k}=\frac{n}{2n+1}+\sum_{k=1}^{n}\frac{1}{(2k)^3-2k}$$

Take the limit of this as $n\to\infty$
Limit of the L.H.S of *

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{k}{n}\right)}.\frac{1}{n}=\int_0^1\frac{dx}{1+x}=log_e|1+x|_0^1=log_e 2$$

and the limit of the first term on the R.H.S of * = $\lim\limits_{n\to\infty}\frac{n}{2n+1}=\lim\limits_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2}$

We thus have $\quad log_e 2=\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{(2k)^3-2k}\quad$ the required result $**$.

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§§ Problem 3: SRMs(1) p4 & (2) p9 and NBSR Vol.I p 7 & Vol.II p13

Show that $\frac{n-1}{n+1}+\frac{n-2}{n+2}+\frac{n-3}{n+3}+\ \dots\dots+\frac{n-n}{n+n}$

$$=2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ \dots\dots\dots\frac{1}{(2n-1)(2n)(2n+1)}\right\}-\frac{n}{2n+1}\qquad *$$

Solution: In the problem No.2 (above), we have the equations (2.2) and (2.3):

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ \dots+\frac{1}{2n}=\frac{n}{2n+1}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ \dots\dots+\frac{1}{(2n-1)(2n)(2n+1)}$$

Multiply both sides by $2n$

$$\frac{2n}{n+1}+\frac{2n}{n+2}+\ \dots\dots+\frac{2n}{2n}(n\ terms)=\frac{2n^2}{2n+1}+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+.\ .\ .\frac{1}{(2n-1)(2n)(2n+1)}\right\}\ \text{-------}\ (3.1)$$

and subtract 1 from each of the n terms on the L.H.S and *n* from the first term of R.H.S. of (3.1)

$L.H.S\ of\ (3.1)\ becomes\ \left(\frac{2n}{n+1}-1\right)+\left(\frac{2n}{n+2}-1\right)+\left(\frac{2n}{n+3}-1\right)+\ \dots\dots\dots+\left(\frac{2n}{2n}-1\right)$

$=\frac{n-1}{n+1}+\frac{n-2}{n+2}+\frac{n-3}{n+3}+\ \dots\dots\dots+\frac{n-n}{n+n}$ = L.H.S. of *and

$R.H.S.of\ (3.1)\ becomes\ \left\{\frac{2n^2}{2n+1}-n\right\}+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\ \dots\dots+\frac{1}{(2n-1)(2n)(2n+1)}\right\}$

$$=-\frac{n}{2n+1}+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\ \dots\dots\dots+\frac{1}{(2n-1)(2n)(2n+1)}\right\}\ =R.H.S\ of\ *$$

The result * is thus established.

§§ Problem 4: SRMs (1) p5 & (2) p10 and NBSR VoI p 8 & also Vol II p14

Show that

$$\frac{1}{n+1}+\frac{1}{n+2}+.\ .\ .\ .+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+.\ .\ .\ .+\frac{2}{(3n)^3-3n}\qquad **$$

and deduce that $\boldsymbol{log_e 3=1+\frac{1}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+.\ .\ .\ .\ .}$ **and so on. $**$**

Solution:

$$R.H.S\ of\ *=1+\sum_{k=1}^{n}\frac{2}{(3k)^3-(3k)}\ =1+2\sum_{k=1}^{n}\frac{1}{(3k-1)(3k)(3k+1)}$$

$$=1+2\sum_{k=1}^{n}\left\{\frac{1}{2(3k-1)}-\frac{1}{3k}+\frac{1}{2(3k+1)}\right\}$$

$$=1+\sum_{k=1}^{n}\left\{\frac{1}{(3k-1)}\right\}-\sum_{k=1}^{n}\left\{\frac{2}{(3k)}\right\}+\sum_{k=1}^{n}\left\{\frac{1}{3k+1}\right\}$$

$$= 1 + \left\{\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ \dots + \frac{1}{3n-1}\right\}$$

$$-\left\{\left[\frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \ \dots + \frac{3}{3n}\right] - \left[\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \dots + \frac{1}{3n}\right]\right\}$$

$$+\left\{\frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \ \dots + \frac{1}{3n+1}\right\}$$

$$= \left\{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ \dots\dots + \frac{1}{3n-1} \ + + \frac{1}{3n} + \frac{1}{3n+1}\right\}$$

$$-\left\{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \dots\dots + \frac{1}{n}\right\}$$

$$= \left\{1 + \frac{1}{2} + \frac{1}{3} + \ \dots\dots + \frac{1}{n}\right\} + \left\{\frac{1}{n+1} + \frac{1}{n+2} + \ \dots + \frac{1}{3n+1}\right\}$$

$$-\left\{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ \dots\dots + \frac{1}{n}\right\}$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ \dots\dots\dots + \frac{1}{3n+1} \ = L.H.S\ of\ *$$

Deduction: We thus have $L.H.S\ of\ *$ in the limit as $n \to \infty$

$$\lim_{n\to\infty} \sum_{k=1}^{2n+1} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{2n+1} \frac{1}{\left(1+\frac{k}{n}\right)} . \frac{1}{n} = \int_0^2 \frac{dx}{1+x} = log_e|1+x|_0^2 = \log_e 3$$

$\therefore \log_e 3 = 1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \frac{2}{9^3-9} + \ \dots\dots\dots + \frac{2}{(3n)^3-3n} + \dots.$ which is the required result. **

§§ Problem 5: SRMs (1) p.5 &(2) p10 and NBSR Vol I p.9 & also Vol II p. 14

$$\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \tan^{-1}\frac{1}{n+3} + \ \dots\dots\dots + \tan^{-1}\frac{1}{3n+1}$$

$$= \tan^{-1} 1 + \tan^{-1}\frac{10}{5.8} + \tan^{-1}\frac{20}{14.35} + \tan^{-1}\frac{30}{29.80} + \ \dots\dots\dots$$

$$+ \tan^{-1}\frac{10n}{(3n^2+2)(9n^2-1)}$$

Corollary:

$$\log_e 3 = \tan^{-1} 1 + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{49} + \tan^{-1}\frac{3}{232} + \tan^{-1}\frac{4}{715} + \ \dots\dots\dots\dots\dots \ **$$

Proof: We recall the following two results on inverse tangent functions:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy} \ \text{ except when } \ xy > 1$$

and

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy} \ \text{ except when } \ -xy > 1$$

$\therefore$ For any positive integer $k \geq 1$, we have

$$\tan^{-1}\frac{1}{3k-1} + \tan^{-1}\frac{1}{3k} + \tan^{-1}\frac{1}{3k+1} - \tan^{-1}\frac{1}{k}$$

$$= \left\{\tan^{-1}\frac{1}{3k-1} + \tan^{-1}\frac{1}{3k+1}\right\} - \left\{\tan^{-1}\frac{1}{k} - \tan^{-1}\frac{1}{3k}\right\}$$

$$= \left\{\tan^{-1}\left(\frac{6k}{9k^2-2}\right)\right\} - \left\{\tan^{-1}\frac{2k}{3k^2+1}\right\}$$

$$= \tan^{-1}\left\{\frac{10k}{(3k^2+1)(9k^2-1)}\right\} \quad \text{------------------------------------ ---------------} \quad (5.1)$$

$$\therefore \ \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{3k-1}\right) + \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{3k}\right) + \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{3k+1}\right) - \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{k}\right)$$

$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{10k}{(3k^2+2)(9k^2-1)}\right) \quad \text{---} \quad (5.2)$$

$$L.H.S\ of\ \ (2) \ = \left\{\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} + \ \dots\dots\dots + \tan^{-1}\frac{1}{3n-1}\right\}$$

$$+\left\{\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{9}+\ \dots\dots\dots+\tan^{-1}\frac{1}{3n}\right\}$$
$$+\left\{\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{10}+\ \dots\dots\dots+\tan^{-1}\frac{1}{3n+1}\right\}$$
$$-\left\{\begin{matrix}\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\quad+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{7}\\ +\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{10}+\ \dots\dots\dots+\tan^{-1}\frac{1}{n}\end{matrix}\right\}$$
$$=-\tan^{-1}1+\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\ .\ .\ .\ .\ .+\tan^{-1}\frac{1}{3n-1}+\tan^{-1}\frac{1}{3n}+\tan^{-1}\frac{1}{3n+1}$$

The equation (2) can now be written after shifting $(-\tan^{-1}1)$ from L.H.S to R.H.S, as

$$\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\ \dots\dots\dots+\tan^{-1}\frac{1}{3n+1}$$
$$=\tan^{-1}1+\tan^{-1}\frac{10}{5.8}+\tan^{-1}\frac{20}{14.35}+\tan^{-1}\frac{30}{29.80}+\ .\ .\ .\ .\ .\ .\ .$$
$$+\tan^{-1}\frac{10n}{(3n^2+2)(9n^2-1)} \qquad *$$

This establishes the result *.

To deduce the Corollary ** :

Let $n\to\infty$ in the identity * , then

$$R.H.S\ of\ *=\ \tan^{-1}1+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}+\ \dots\dots\dots\dots$$
$$L.H.S=\lim_{n\to\infty}\sum_{k=n+1}^{3n+1}\tan^{-1}\frac{1}{n+k}$$
$$=\lim_{n\to\infty}\sum_{k=n+1}^{3n+1}\frac{1}{n+k}=\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{1}{1+\frac{k}{n}}\cdot\frac{1}{n}$$
$$=\int_{x=0}^{2}\frac{dx}{1+x}=\log_e 3$$ Hence the Corollary **.

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§§ Problem 6: SRMs(3)p.4 NBSR Vol. II p364.

$$\frac{1}{\boldsymbol{logx}}+\frac{1}{1-x}=\frac{1}{2(1+\sqrt{x})}+\frac{1}{4\left(1+\sqrt[4]{x}\right)}+\frac{1}{8\left(1+\sqrt[8]{x}\right)}+\text{- - - -- - - - - -} *$$

Solution:

Consider the identity

$$\frac{1}{1-a^2}=\frac{1}{2}\left\{\frac{1}{(1+a)}+\frac{1}{(1-a)}\right\} \qquad \text{- - - - - - - - - - - -}\ (6.\ 1\ )$$

$$\therefore\ \frac{1}{1-x}=\frac{1}{1-(\sqrt{x})^2}=\frac{1}{2}\left\{\frac{1}{1+x^{1/2}}+\frac{1}{1-x^{1/2}}\right\}$$ Here a = $x^{1/2}$

$$=\frac{1}{2\left(1+x^{1/2}\right)}+\frac{1}{2\left(1-x^{1/2}\right)}$$
$$=\frac{1}{2\left(1+x^{1/2}\right)}+\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{\left(1+x^{1/2^2}\right)}+\frac{1}{\left(1-x^{1/2^2}\right)}\right]\right\}$$ Here a = $x^{1/2^2}$

$$=\frac{1}{2\left(1+x^{1/2}\right)}+\frac{1}{2^2}\frac{1}{\left(1+x^{1/2^2}\right)}+\frac{1}{2^2}\frac{1}{\left(1-x^{1/2^2}\right)}$$
$$=\frac{1}{2\left(1+x^{1/2}\right)}+\frac{1}{2^2}\frac{1}{\left(1+x^{1/2^2}\right)}+\frac{1}{2^3}\frac{1}{\left(1+x^{1/2^3}\right)}+\frac{1}{2^4}\frac{1}{\left(1+x^{1/2^4}\right)}+\frac{1}{2^4}\frac{1}{\left(1-x^{1/2^4}\right)}$$

Thus by repeated application of the identity (1) n times, we get

$$\frac{\mathbf{1}}{\mathbf{1}-\mathrm{x}}=\sum_{\mathrm{k}=\mathbf{1}}^{\mathrm{n}}\frac{\mathbf{1}}{\mathbf{2}^k\left(\mathbf{1}+x^{\mathbf{1}/\mathbf{2}^k}\right)}+\frac{\mathbf{1}}{\mathbf{2}^n\left\{\mathbf{1}-x^{\mathbf{1}/\mathbf{2}^n}\right\}} \qquad \text{- - - - - - - - - - - - - - - - - -}\ (6.\ 2\ )$$

The result (*) can be obtained by taking the limit of the result (2) as n → ∞.

Now $$\lim_{x\to\infty}\frac{1}{2^n\left\{1-x^{1/2^n}\right\}} = \lim_{\theta\to 0}\frac{\theta}{\left\{1-x^{\theta}\right\}}$$ where $\theta=\frac{1}{2^n}$

$$\lim_{\theta\to 0}\frac{\theta}{\left\{1-e^{\theta\log_e x}\right\}} = -\frac{1}{\log_e x} \qquad \text{-------------- (6.3)}$$

As n → ∞, the result (2) becomes

$$\frac{1}{1-x}=\sum_{k=1}^{\infty}\frac{1}{2^k\left(1+x^{1/2^k}\right)}-\frac{1}{\log_e x} \qquad \frac{1}{\log_e x}+\frac{1}{1-x}=\sum_{k=1}^{\infty}\frac{1}{2^k\left(1+x^{1/2^k}\right)}$$

Hence $$\frac{1}{\log_e x}+\frac{1}{1-x}=\sum_{k=1}^{\infty}\frac{1}{2^k\left(1+x^{1/2^k}\right)}$$

Note: $$\lim_{\theta\to 0}\frac{\theta}{\left\{1-e^{a\theta}\right\}} = \lim_{\theta\to 0}\frac{\theta}{\left\{1-\left[1+\frac{a\theta}{1!}+\frac{a^2\theta^2}{2!}+\frac{a^3\theta^3}{3!}+\dots\dots\dots\right]\right\}}$$

$$= \lim_{\theta\to 0}\frac{\theta}{-a\theta\left\{1+\frac{a\theta}{2!}+\frac{(a\theta)^2}{3!}+\dots\dots\right\}} = -\frac{1}{a}$$

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§§ Problem 7: NBSR Vol. II p 364

$$\frac{1}{\log x}+\frac{1}{1-x} = \frac{2+\sqrt[3]{x}}{3\left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)}+\frac{2+\sqrt[9]{x}}{9\left(1+\sqrt[9]{x}+\sqrt[9]{x^2}\right)}+ \text{- ------------- *}$$

Solution:

Consider the identity

$$\frac{1}{1-a^3}=\frac{1}{(1+a+a^2)(1-a)}=\frac{1}{3}\cdot\frac{2+a}{(1+a+a^2)}+\frac{1}{3}\cdot\frac{1}{(1-a)} \qquad \text{-------------- (7.1)}$$

Using this identity, we write

$$\frac{1}{1-x} = \frac{1}{1-\left(x^{1/3}\right)^3} = \frac{2+x^{1/3}}{3\left\{1+x^{1/3}+\left(x^{1/3}\right)^2\right\}}+\frac{1}{3}\left[\frac{1}{3}\cdot\frac{2+x^{1/3^2}}{\left\{1+x^{1/3}+\left(x^{1/3^2}\right)^2\right\}}+\frac{1}{3}\cdot\frac{1}{\left(1-x^{1/3^2}\right)}\right]$$

$$=\frac{2+x^{1/3}}{3\left\{1+x^{1/3}+\left(x^{1/3}\right)^2\right\}}+\frac{2+x^{1/3^2}}{3^2\left\{1+x^{1/3}+\left(x^{1/3^2}\right)^2\right\}}+\frac{1}{3^2}\left[\frac{1}{3}\cdot\frac{2+x^{1/3^3}}{\left\{1+x^{1/3^3}+\left(x^{1/3^3}\right)^2\right\}}+\frac{1}{3}\cdot\frac{1}{\left(1-x^{1/3^3}\right)}\right]$$

By repeated application of the identity (7.1), we get

$$\frac{1}{1-x}=\sum_{k=1}^{n}\frac{2+x^{1/3^k}}{3^k\left\{1+x^{1/3^k}+\left(x^{1/3^k}\right)^2\right\}}+\frac{1}{3^n\left\{1-x^{1/3^n}\right\}} \qquad \text{---------------- (7.2)}$$

Taking the limit of the above result as n → ∞ , we get

$$\frac{1}{1-x}=\sum_{k=1}^{n}\frac{2+x^{1/3^k}}{3^k\left\{1+x^{1/3^k}+\left(x^{1/3^k}\right)^2\right\}}+\lim_{x\to\infty}\frac{1}{3^n\left\{1-x^{1/3^n}\right\}} \qquad \text{------------ (7. 3)}$$

The second term on the R.H.S. of (7.3) = $\lim\limits_{x\to\infty} \dfrac{1}{3^n\left\{1-x^{1/3^n}\right\}}$ = $\lim\limits_{\theta\to 0} \dfrac{\theta}{\left\{1-x^{\theta}\right\}}$ where $\theta = \frac{1}{3^n}$

$= \dfrac{1}{-\log x}$

$$\therefore\ \frac{1}{\log x} + \frac{1}{1-x} = \frac{2+\sqrt[3]{x}}{3\left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)} + \frac{2+\sqrt[9]{x}}{9\left(1+\sqrt[9]{x}+\sqrt[9]{x^2}\right)} + \text{- - -} + \frac{2+x^{1/3^n}}{3^n\left\{1+x^{1/3^n}+\left(x^{1/3^n}\right)^2\right\}} + \text{- - - - - - - - - - - -}$$

This establishes the identity.

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§§ Problem 8: Lost NBSR p. 333

(1+ x)(1+ x^2)(1+ x^3)(1+ x^4) . . . . . . and so on = $\dfrac{1}{(x-1)(1-x^3)(1-x^3)(1-x^3)\ldots\ldots\ldots..}$ *

Note: The L.H.S. of * is the product of all the factors of the type 1 + x^n, n = 1, 2, 3, 4, . . . while the R.H.S. is the reciprocal of the product of the factors of the type 1 – x^n where n takes successively all odd numbers i.e., 1, 3, 5, 7, ...

Solution :

Since 1 + x^n = $\dfrac{1-x^{2n}}{1-x^n}$, $n = 1, 2, 3, \ldots\ldots\ldots..$

L.H.S. of * = (1+ x)(1+ x^2)(1+ x^3)(1+ x^4) . . . .

$$= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdot\frac{1-x^{10}}{1-x^5}\cdot\frac{1-x^{12}}{1-x^6}\ldots\ldots.. \text{ and so on. ------------ (1)}$$

The numerator on the R.H.S. of (1) is the product of all the factors of the type 1 – x^{2n}, n =1, 2, 3, 4, . . . and all these factors can be seen in the denominator at alternate places in the product. Cancelling all such factors (found in the numerator and denominator of (1)), we get

L.H.S. of * = $\dfrac{1}{(x-1)(1-x^3)(1-x^3)(1-x^3)\ldots\ldots\ldots..}$ = R.H.S. of * . Hence the result *.

§§ Problem 9: SRMs (1) p.49 and NBSR Vol.I p. 97

$$\frac{1}{x-1}+\frac{1}{x^2-1}+\frac{1}{x^3-1}+\frac{1}{x^4-1}+\ldots\ldots = \frac{x+1}{x(x-1)}+\frac{x^2+1}{x^4(x^2-1)}+\frac{x^3+1}{x^9(x^3-1)}+\frac{x^4+1}{x^{16}(x^4-1)}+\ldots\ldots$$

Solution:

The nth term on L.H.S. of * = $\frac{1}{x^n - 1}$ and that on the R.H.S. = $\frac{x^n + 1}{x^{(n^2)}(x^n - 1)}$, n = 1, 2, 3, 4, ... Each of terms on the L.H.S. of * are split up into parts as shown in the adjoining table and the parts are grouped into blocks indicated. All these parts are added block wise to get the result: R.H.S. of *. In the block, barring the first member, we notice Geometric Progressions whose first terms are $\frac{1}{x^2}, \frac{1}{x^6}, \frac{1}{x^{12}}, \frac{1}{x^{20}}$, ...and common differences $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}$. . . . . respectively. Further the number of members is infinitely large. Though not stated explicitly S.R. in the problem, it is presumed that $|x|$ > 1, an essential condition for the computation of sum to infinity of each of the above geometric series : (First term) x (1 – common ratio)

Now

Sum of the terms in block (1)= $\frac{1}{x-1} + \left\{ \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + ... \right\} = \frac{1}{x-1} + \frac{1}{x(x-1)} = \frac{x+1}{x(x-1)}$

S.No	Term	Split up form of the term					
1	$\frac{1}{x-1} =$	$\frac{1}{x-1}$ +					
2	$\frac{1}{x^2-1} =$	$\frac{1}{x^2}$ +	$\frac{1}{x^2(x^2-1)}$ +				
3	$\frac{1}{x^3-1} =$	$\frac{1}{x^3}$ +	$\frac{1}{x^6}$ +	$\frac{1}{x^6(x^3-1)}$ +			
4	$\frac{1}{x^4-1} =$	$\frac{1}{x^4}$ +	$\frac{1}{x^8}$ +	$\frac{1}{x^{12}}$ +	$\frac{1}{x^{12}(x^4-1)}$ +		
5	$\frac{1}{x^5-1} =$	$\frac{1}{x^5}$ +	$\frac{1}{x^{10}}$ +	$\frac{1}{x^{15}}$ +	$\frac{1}{x^{20}}$ +	$\frac{1}{x^{20}(x^5-1)}$ +	
6	$\frac{1}{x^6-1} =$	$\frac{1}{x^6}$ +	$\frac{1}{x^{12}}$ +	$\frac{1}{x^{18}}$ +	$\frac{1}{x^{24}}$ +	$\frac{1}{x^{30}}$ +	$\frac{1}{x^{30}(x^6-1)}$ +
7	$\frac{1}{x^7-1} =$	$\frac{1}{x^7}$ +	$\frac{1}{x^{14}}$ +	$\frac{1}{x^{21}}$ +	$\frac{1}{x^{28}}$ +	$\frac{1}{x^{35}}$ +	$\frac{1}{x^{42}}$ +
Block Number:		(1)	(2)	(3)	(4)	(5)	(6)

Sum of the terms in block (2) = $\frac{1}{x^2(x^2-1)} + \left\{ \frac{1}{x^6} + \frac{1}{x^8} + \frac{1}{x^{10}} + \frac{1}{x^{12}} + ... \right\}$

$= \frac{1}{x^2(x^2-1)} + \frac{1}{x^4} \cdot \frac{1}{(x^2-1)} = \frac{x^2+1}{x^4(x^2-1)}$

Sum of the terms in block (3) = $\frac{1}{x^6(x^3-1)} + \frac{1}{x^9(x^3-1)} = \frac{x^3+1}{x^9(x^3-1)}$

Similarly, the sum in the block (4) = $\frac{1}{x^{12}(x^4-1)}+\frac{1}{x^{16}(x^4-1)} = \frac{x^4+1}{x^{16}(x^4-1)}$

that in the block (5) = $\frac{1}{x^{20}(x^5-1)}+\frac{1}{x^{25}(x^5-1)} = \frac{x^5+1}{x^{25}(x^5-1)}$ . . . . and so on.

In general, sum of the terms in the nth block = $\frac{x^m+1}{x^{n^2}(x^n-1)}$

Adding the terms in all the boxes, we get

L.H.S. of * = $\frac{x+1}{x(x-1)}+\frac{x^2+1}{x^4(x^2-1)}+\frac{x^3+1}{x^9(x^3-1)}+\frac{x^4+1}{x^{16}(x^4-1)}+\ldots\ldots\ldots$ and so on.

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§§ Problem 10: SRMs (1) p.50 and NBSR Vol. I p. 99

$$\frac{r}{1-ax}+\frac{r^2}{1-ax^2}+\frac{r^3}{1-ax^3}+\frac{r^4}{1-ax^4}+\ldots\ldots \text{ to n terms}$$

$$= \left\{\frac{arx}{1-ax}+\frac{(arx^2)^2}{1-ax^2}+\frac{(arx^3)^3}{1-ax^3}+\frac{(arx^4)^4}{1-ax^4}+\ldots\ldots\ldots\ldots\text{to n terms}\right\} +$$

$$\left\{\frac{r-r^{n+1}}{1-r}+a.\frac{(rx)^2-(rx)^{n+1}}{1-rx}+a^2.\frac{(rx^2)^3-(rx^2)^{n+1}}{1-rx^2}+a^3.\frac{(rx^3)^4-(rx^3)^{n+1}}{1-rx^3}\ldots\ldots+\text{to n terms}\right\} \quad *$$

Further, when the number of terms is infinitely large (i.e., as n $\to \infty$) the result is

$$\frac{r}{1-ax}+\frac{r^2}{1-ax^2}+\frac{r^3}{1-ax^3}+\frac{r^4}{1-ax^4}+\ldots\ldots\ldots \text{ and so on (to infinite number of terms).}$$

$$= \left\{\frac{arx}{1-ax}+\frac{(arx^2)^2}{1-ax^2}+\frac{(arx^3)^3}{1-ax^3}+\frac{(arx^4)^4}{1-ax^4}+\ldots\ldots\ldots\ldots\textit{and so on}\right\}$$

$$\left\{\frac{r}{1-r}+\frac{a.(rx)^2}{1-rx}+\frac{a^2.(rx^2)^3}{1-rx^2}+\frac{a^3.(rx^3)^4}{1-rx^3}+\frac{a^4.(rx^4)^5}{1-rx^4}+\ldots\ldots\ldots\right\} \quad **$$

Solution:

The L.H.S. contains n terms and a typical term = $\dfrac{r^k}{1-ax^k}$ **,** k =1, 2, 3, 4, ...

The terms on the L.H.S. of * can be split up as shown in the following table.
These terms can be added column wise (indicated by the blocks in the table.)

S.No	Term	Split up form of the term										
1	$\dfrac{r}{1-ax}=$	$\dfrac{arx}{1-ax}$	+	r								
2	$\dfrac{r^2}{1-ax^2}=$	$\dfrac{(arx^2)^2}{1-ax^2}$	+	r^2	+	$a r^2 x^2$						
3	$\dfrac{r^3}{1-ax^3}=$	$\dfrac{(arx^3)^3}{1-ax^3}$	+	r^3	+	$a r^3 x^3$	+	$a^2 r^6 x^6$				
4	$\dfrac{r^4}{1-ax^4}=$	$\dfrac{(arx^4)^4}{1-ax^4}$	+	r^4	+	$a r^4 x^4$	+	$a^2 r^4 x^8$	+	$a^3 r^4 x^{12}$		
5	$\dfrac{r^5}{1-ax^5}=$	$\dfrac{(arx^5)^5}{1-ax^5}$	+	r^5	+	$a r^5 x^5$	+	$a^2 r^5 x^{10}$	+	$a^3 r^5 x^{15}$	+	$a^4 r^5 x^{20}$ +
	………	………		..		……		………..		………		……………
k	$\dfrac{r^k}{1-ax^k}=$	$\dfrac{(arx^k)^k}{1-ax^k}$	+	r^k	+	$a r^k x^k$	+	$a^2 r^k (x^2)^k$	+	$a^3 r^k (x^3)^k$	+	$a^4 r^k (x^4)^k$ + K = 1,2,3,….n
Block Number:		(0)		(1)		(2)		(3)		(4)		(5)

Sum of the terms block No. (0)

$$= \frac{arx}{1-ax}+\frac{(arx^2)^2}{1-ax^2}+\frac{(arx^3)^3}{1-ax^3}+\frac{(arx^4)^4}{1-ax^4}+\ldots\ldots\ldots+\frac{(arx^n)^n}{1-ax^n} \text{ (n terms)}$$

= First { } bracket on the R.H.S. of *

Sum in the block No. (1) = $r + r^2 + r^3 + r^4 + \ldots\ldots\ldots$ n terms = $r\left\{\dfrac{1-r^n}{1-r}\right\} = \dfrac{r-r^{n+1}}{1-r}$

Sum in the block No. (2) = $ar^2x^2 + ar^3x^3 + ar^4x^4 + \ldots\ldots +$ (n - 1) terms

$$= ar^2x^2\left\{\frac{1-(rx)^{n-1}}{1-rx}\right\} = a\left\{\frac{(rx)^2-(rx)^{n+1}}{1-rx}\right\}$$

Sum in the block No. (3)
= $a^2r^3x^6 + a^2r^4x^8 + a^2r^5x^{10} + \ldots\ldots +$ (n - 2) terms

$$= a^2\left(rx^2\right)^3\left\{\frac{1-\left(rx^2\right)^{n-2}}{1-(rx^2)}\right\} = a^2\left\{\frac{\left(rx^2\right)^3-\left(rx^2\right)^{n+1}}{1-rx^2}\right\}$$

Sum in the block No. (4)
= $a^3r^4x^{12} + a^3r^5x^{15} + a^3r^7x^{18} + \ldots\ldots +$ (n - 3) terms

$$= a^2\left(rx^3\right)^4\left\{\frac{1-\left(rx^3\right)^{n-2}}{1-(rx^3)}\right\} = a^3\left\{\frac{\left(rx^3\right)^4-\left(rx^3\right)^{n+1}}{1-rx^3}\right\} \ldots\ldots \text{ and so on.}$$

Adding all these we set the second { } bracket of the R.H.S. OF *.
Hence the result *.

To establish the identity * *

It is presumed that $\left|rx^k\right| < 1$ for k = 1, 2, 3, 4, ... n. (a condition not stated explicitly by S.R.) . Then as $n \to \infty$ the geometric series in each of the blocks is the sum to infinity of a geometric series whose common ratios are (rx^k , k = 1, 2, 3, ………..) less than 1.

Hence as $n \to \infty$, * reduces to * *.

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§§ Problem 11: SPMs (II) p 298 and NBSR Vol.II p 355, BB V 487

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} + \frac{x^9}{1-x^9} + \text{- - - - - - - - - - - - - - -}$$
$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \frac{x^{10}}{1-x^4} + \frac{x^{15}}{1-x^5} + \text{- - - - - - - - - - - - - -} \qquad *$$

Note : The indices of x in the terms of the L.H.S. are all odd numbers { 2n + 1, n = 0, 1, 2, 3, - - - - } while the indices of x in the numerators of the terms of the R.H.S. are the Triangular Numbers : { $T_n = \frac{n(n+1)}{2}$, n = 1, 2, 3, - - - - - - - }.

Solution:

L.H.S. of * = $x(1-x)^{-1} + x^3(1-x^3)^{-1} + x^5(1-x^5)^{-1} + x^7(1-x^7)^{-1}$ + - - - - and so on.

$$\begin{array}{lllllllllllllll}
= \mathbf{x}\,.\ \{ & \mathbf{1} & + & \mathbf{x} & + & \mathbf{x^2} & + & \mathbf{x^3} & + & \mathbf{x^4} & + & \mathbf{x^5} & + & \ldots\ldots\ldots\ldots (1) & \}\\
+\ \mathbf{x^3}.\ \{ & \mathbf{1} & + & \mathbf{x^3} & + & \mathbf{x^6} & + & \mathbf{x^9} & + & \mathbf{x^{12}} & + & \mathbf{x^{15}} & + & \ldots\ldots\ldots\ldots (3) & \}\\
+\ \mathbf{x^5}.\ \{ & \mathbf{1} & + & \mathbf{x^5} & + & \mathbf{x^{10}} & + & \mathbf{x^{15}} & + & \mathbf{x^{20}} & + & \mathbf{x^{25}} & + & \ldots\ldots\ldots\ldots (5) & \}\\
+\ \mathbf{x^7}.\ \{ & \mathbf{1} & + & \mathbf{x^7} & + & \mathbf{x^{14}} & + & \mathbf{x^{21}} & + & \mathbf{x^{28}} & + & \mathbf{x^{35}} & + & \ldots\ldots\ldots\ldots (7) & \}\\
+\ \mathbf{x^9}.\ \{ & \mathbf{1} & + & \mathbf{x^9} & + & \mathbf{x^{18}} & + & \mathbf{x^{27}} & + & \mathbf{x^{36}} & + & \mathbf{x^{45}} & + & \ldots\ldots\ldots\ldots (9) & \}\\
+\ \mathbf{x^{11}}.\{ & \mathbf{1} & + & \mathbf{x^{11}} & + & \mathbf{x^{22}} & + & \mathbf{x^{33}} & + & \mathbf{x^{44}} & + & \mathbf{x^{55}} & + & \ldots\ldots\ldots\ldots (11) & \}\\
 & (2) & & (4) & & (6) & & (8) & & (10) & & \text{and so on} & & &
\end{array}$$

Rearranging the terms as indicated in the boxes shown above,

L.H.S. of * = $\mathbf{x}\,\{\,\mathbf{1 + x + x^2 + x^3 + x^4 + x^5 +}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\}\ \mathbf{(1)}$

$+\ \mathbf{x^3}\,\{\,\mathbf{1 + x^2 + x^4 + x^6 + x^8 + x^{10} +}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\}\ \mathbf{(2)}$

$+\ \mathbf{x^6}\,\{\,\mathbf{1 + x^3 + x^6 + x^9 + x^{12} + x^{15} +}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\}\ \mathbf{(3)}$

$+\ \mathbf{x^{10}}\,\{\,\mathbf{1 + x^3 + x^6 + x^9 + x^{12} + x^{15} +}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\}\ \mathbf{(4)}$

$+\ \mathbf{x^{15}}\,\{\,\mathbf{1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25} +}\ \ldots\ldots\ldots\ldots\ldots\ldots\}\ \mathbf{(5)}$

$+\ \mathbf{x^{21}}\,\{\,\mathbf{1 + x^6 + x^{12} + x^{18} + x^{24} + x^{30} +}\ \ldots\ldots\ldots\ldots\ldots\ldots\}\ \mathbf{(6)}$

+ . . . . and so on.

$$= x(1-x)^{-1} + x^3(1-x^2)^{-1} + x^6(1-x^3)^{-1} + x^{10}(1-x^4)^{-1}\ldots\ldots\ldots + x^{15}(1-x^5)^{-1} + \text{- - - -}$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \frac{x^{10}}{1-x^4} + \frac{x^{15}}{1-x^5} + \text{- - - - - - - - - - - -} = \text{R.H.S. of } *$$

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Problem 12: Lost NBSR p. 333

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\ \ldots\ldots\ldots = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots\ldots\ldots} \qquad *$$

Solution:

Note: The index of x in each factor of the L.H.S is a natural number:1, 2, 3, 4 .... and that each factor of the R.H.S is an odd number:1, 3, 5, 7 ....

L.H.S. of * $= (1 + x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)$ - - - - - - - - - - - - - - - - - - -

$$= \frac{(1-x^2)}{1-x}\cdot\frac{(1-x^4)}{1-x^2}\cdot\frac{(1-x^6)}{1-x^3}\cdot\frac{(1-x^8)}{1-x^4}\cdot\frac{(1-x^{10})}{1-x^5}\cdot\frac{(1-x^{12})}{1-x^6}\cdot\frac{(1-x^{14})}{1-x^8}\ldots\ldots\ldots$$

Cancelling the terms $(1 - x^2)$, $(1 - x^4)$, $(1 - x^6)$, ... and so on $(1 - x^{2n})$ accruing in numerator and denominator

$$= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^9}\ldots\ldots\ldots$$

= R.H.S. of *.

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----------------- (To be continued ……)

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