International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 1, Issue 2, October - 2013, PP 134-143
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Prime and Semiprime Bi-Ideals of So-Rings

M. Srinivasa Reddy
Assistant Professor
Department of S & H
D. V. R & Dr. H. S. MIC College of Technology
Kanchikacherla, A.P, India
maths4444@gmail.com

Dr. V. Amarendra Babu
Assistant Professor
Department of Mathematics
Acharya Nagarjuna University
Guntur, A.P, India
amarendravelisela@ymail.com

Dr. P. V. Srinivasa Rao
Associate Professor
Department of S & H
D. V. R & Dr. H. S. MIC College of Technology
Kanchikacherla, A.P, India
srinu_fu2004@yahoo.co.in

Received: 16-08-2013 **Revised: 07-09-2013** **Accepted: 10-09-2013**

Abstract: *The partial functions under disjoint-domain sums and functional composition do not form a field, and thus conventional linear algebra is not applicable. However they can be regarded as a so-ring, an algebraic structure possessing a natural partial ordering, an infinitary partial addition and a binary multiplication, subject to a set of axioms. In this paper the notions of prime and semiprime bi-ideals in so-rings are introduced and obtained some characteristics of prime and semiprime bi-ideals of so-rings.*

Keywords: *Prime bi-ideal, semiprime bi-ideal, p-system, m-system, multiplicatively regular, irreducible and strongly irreducible bi-ideals.*

2010 Mathamatics Subject Classification: *16Y60*

1. INTRODUCTION

The study of $pfn(D, D)$ (the set of all partial functions of a set D to itself), $Mfn(D, D)$ (the set of all multi functions of a set D to itself) and $Mset(D, D)$ (the set of all total functions of a set D to the set of all finite multi sets of D) play an important role in the theory of computer science, and to abstract these structures Manes and Benson[5] introduced the notion of sum ordered partial semirings(so-rings). Motivated by the work done in partially-additive semantics by Arbib, Manes [3] and in the development of matrix theory of so-rings by Martha E. Streenstrup[6]. G. V. S. Acharyulu[1] in 1992 studied conditions under which an arbitrary so-ring becomes a $pfn(D, D)$, $Mfn(D, D)$ and $Mset(D, D)$. Continuing this study, P. V. Srinivasa Rao[8] in 2011 developed the ideal theory for so-rings. In this paper we introduce the notions of prime and semiprime bi-ideals and observe the characteristics of prime radical interms of semiprime bi-ideals.

2. PRELIMINARIES

In this section we collect important definitions, results and examples which were already proved for our use in the next sections.

2.1 Definition. [5] A *partial monoid* is a pair (M,Σ) where M is a non empty set and $\sum$ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:

(1) Unary Sum Axiom: If $(x_i : i \in I)$ is a one element family in M and $I = \{ j \}$, then $\sum(x_i : i \in I)$ is defined and equals x_j.

(2) Partition - Associativity Axiom: If $(x_i : i \in I)$ is a family in M and If $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J and ($\sum(x_i : i \in I_j) : j \in J$) is summable. We write $\sum(x_i : i \in I) = \sum(\sum(x_i : i \in I_j) : j \in J)$.

2.2 Definition. [5] The *sum ordering* $\leq$ on a partial monoid (M,Σ) is the binary relation $\leq$ such that $x \leq y$ if and only if there exists a h in M such that $y = x + h$, for $x, y \in M$.

2.3 Definition. [5] A *partial semiring* is a quadruple $(R,\Sigma,\cdot,1)$, Where (R,Σ) is a partial monoid with partial addition $\sum$, $(R,\cdot,1)$ is a monoid with multiplicative operation '$\cdot$' and unit '1', and the additive and multiplicative structures obey the following distributive laws:

If $\sum(x_i : i \in I)$ is defined in R, then for all y in R, $\sum(y \cdot x_i : i \in I)$ and $\sum(x_i \cdot y : i \in I)$ are defined and $y \cdot [\sum_i x_i] = \sum_i (y \cdot x_i), [\sum_i x_i] \cdot y = \sum_i (x_i \cdot y)$.

2.4 Definition. [5] A *sum-ordered partial semiring* (or *so-ring* for short), is a partial semiring in which the sum ordering is a partial ordering.

2.5 Definition. [1] Let R be so-ring. A subset N of R is said to be an *ideal* of R if the following are satisfied:

(I_1) if $(x_i : i \in I)$ is a summable family in R and $x_i \in N$ for every $i \in I$ then $\sum x_i \in N$,

(I_2) if $x \leq y$ and $y \in N$ then $x \in N$, and

(I_3) if $x \in N$ and $r \in R$ then $xr, rx \in N$.

2.6 Definition. [2] A subset N of a so-ring R is said to be a *bi-ideal* of R if the following are satisfied

(B_1) if $(x_i : i \in I)$ is a summable family in R and $x_i \in N$ for every $i \in I$ then $\sum_i x_i \in N$,

(B_2) if $x \leq y$ and $y \in N$ then $x \in N$, and

(B_3) if $x, y \in N$ and $r \in R$ then $xry \in N$.

Note that every ideal is a bi-ideal. The following is an example of a so-ring in which bi-ideal is not an ideal.

2.7 Example. [2] Consider the so-ring $\mathrm{N} = \mathrm{N} \cup \{0\}$ the set of all natural numbers with '0'. Take $R = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} / a,b,c,d \in \mathrm{N} \right)$. Then R is a so-ring with respect to matrix addition and matrix multiplication. Now $B = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} / x \in \mathrm{N} \right\}$ is a bi-ideal but not an ideal of R.

2.8 Example. [2] Consider the so-ring $R = \{0, u, v, x, y, 1\}$ with $\sum$ defined on R by

$$\sum_i x_i = \begin{cases} x_j & \text{if } x_i = 0 \quad \forall i \neq j, \quad \text{for some} \quad j, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

And ‘ · ’ defined by the following table:

·	0	u	v	x	y	1
0	0	0	0	0	0	0
u	0	u	0	0	0	u
v	0	0	v	0	0	v
x	0	0	0	0	0	x
y	0	0	0	0	0	y
1	0	u	v	x	y	1

Then for bi-ideals $\{0,x,y\}$, $\{0,u,x\}$ of R, $\{0,x,y\}\cap\{0,u,x\}=\{0,x\}$ whereas $\{0,x,y\}\{0,u,x\} = \{0\}$.

2.9 Example. [2] Consider the so-ring $R= \{0, a, b, c, d, 1\}$ with $\sum$ on R defined by

$$\sum_i x_i = \begin{cases} x_{j,} & \text{if} \quad x_i = 0 \,\forall i \neq j, \quad \text{for some} \quad j, \\ d, & \text{if} \quad x_j = a, x_k = b \quad \text{or} \quad x_j = b, x_k = c \quad \text{for some} \quad j, k, x_i = 0 \quad \forall i \neq j, k, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

And ‘·’ defined by

$$x \cdot y = \begin{cases} 0, & \text{if} \quad x \neq 1, y \neq 1, \\ x, & \text{if} \quad y = 1, \\ y & \text{if} \quad x = 1. \end{cases}$$

Then the bi-ideals of R are $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a, b, c, d\}$, R. Now $\{0, a\}\cup\{0, b\}=\{0, a, b\}$ is not a bi-ideal of R, since $a + b = d$ which is not in $\{0, a, b\}$.

2.10 Definition. [8] A proper ideal P of so-ring R is said to be *prime* if and only if for any ideals A,B of R, $AB \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$.

2.11 Definition. [8] An element a of a partial semiring R is said to be *multiplicatively regular* if and only if there exists a $b \in R$ such that $aba = a$.

2.12 Definition.[8] A partial semiring R is said to be *multiplicatively regular* if and only if each element of R is multiplicatively regular.

3. PRIME BI-IDEALS

In this section, we define a prime bi-ideal of a so-ring R and characterize the prime radical interms of prime bi-ideals of R.

3.1 Definition. Let R be a so-ring and a in R. Then the *principal ideal generated by a* is

$$< a > = \left\{x \in R / x \leq \sum a + ara\,, \;\; a \in R\right\}$$

3.2 Definition. A proper bi-ideal of a so-ring R is said to be *prime* if and only if for any bi-ideals A, B of R, $ARB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

3.3 Example. Consider the so-ring R = [0,1]. Since for any bi-ideals [0, x], [0, y] and [0, z] of R, $[0, x]R[0, y] \subseteq [0, z]$ implies that $[0, x] \subseteq [0, z]$ or $[0, y] \subseteq [0, z]$, every bi-ideal of R is a prime bi-ideal of R.

3.4 Theorem. If P is a proper bi-ideal of a complete so-ring R then the following are equivalent:

(i) P is prime, and

(ii) $\{arb/r \in R\} \subseteq P \Rightarrow a \in P$ *or* $b \in P$

Proof: (i) $\Rightarrow$ (ii): Suppose P is prime and take $P' = \{arb / r \in R\}$. Suppose $P' \subseteq P$ and take $A = \langle a \rangle$, $B = \langle b \rangle$. Let $x \in ARB$. Then $x \leq \sum_i a_i r_i b_i$ for $a_i \in \langle a \rangle$, $b_i \in \langle b \rangle$, $r_i \in R$. $\Rightarrow$For any $i \in I$, $a_i \leq \sum a + as_1a$ and $b_i \leq \sum b + bs_2b$ where s_1, $s_2 \in R$.

$$\Rightarrow x \leq \sum_i (\sum a + as_1a) r_i (\sum b + bs_2b)$$
$$= \sum_i [(\sum a) r_i (\sum b) + (\sum a) r_i (bs_2b) + (as_1a) r_i (\sum b) + (as_1a) r_i (bs_2b)]$$
$$= \sum_i [\sum \sum ar_ib + \sum a(r_ibs_2)b + \sum a(s_1ar_i)b + a(s_1ar_ibs_2)b]$$
$$= \sum_i \sum \sum ar_ib + \sum_i \sum a(r_ibs_2)b + \sum_i \sum a(s_1ar_i)b + \sum_i a(s_1ar_ibs_2)b.$$

Since $P' \subseteq P$ and P is a bi-ideal of R, we have $x \in P$. Therefore $ARB \subseteq P \Rightarrow A = \langle a \rangle \subseteq P$ or $B = \langle b \rangle \subseteq P$. Hence $a \in P$ or $b \in P$.

(ii) $\Rightarrow$(i): Suppose $P' = \{arb / r \in R\} \subseteq P \Rightarrow a \in P$ or $b \in P$. Let A, B be bi-ideals of R such that $ARB \subseteq P$ and suppose that $A \not\subset P$. Then $\exists x \in A \ni x \notin P$. For any $y \in B$, $\{xry / r \in R\} \subseteq ARB \subseteq P$. $\Rightarrow x \in P$ or $y \in P$. $\Rightarrow y \in P \forall y \in B$. Therefore $B \subseteq P$. Hence P is a prime ideal.

3.5 Definition. A so-ring R is said to be *prime* if and only if < 0 > is a prime bi-ideal. $Pfn(D, D), Mfn(D, D)$ and $Mset(D, D)$ are prime so-rings for any non empty set D. It may be noted that the so-ring R considered in the example 2.8 is not a prime so-ring.

3.6 Lemma. A so-ring R is prime if and only if $1 \neq 0$ and for each pair of nonzero elements $a, b \in R$, there exists r in R such that $arb \neq 0$.

3.7 Definition. A non empty subset A of a so-ring R is said to be an *m-system* if and only if for any $a, b \in A$, there exists $r \in R \ni arb \in A$.

3.8 Example. Consider the so-ring R as in the example 2.8. Then set $\{0, u, v\}$ is an m-system of R.

3.9 Theorem. A proper bi-ideal P of a complete so-ring R is prime if and only if $R \setminus P$ is an m-system.

Proof: A bi-ideal P of R is prime $\Leftrightarrow$ $\{arb / r \in R\} \subseteq P$ then $a \in P$ or $b \in P$ (Since by the theorem 3.4)$\Leftrightarrow$ $a \notin P$ and $b \notin P$ then $\{arb / r \in R\} \not\subset P \Leftrightarrow$for every $a, b \in R \setminus P, \exists r \in R \ni arb \in R \setminus P \Leftrightarrow R \setminus P$ is an m-system.

3.10 Theorem. A bi-ideal B of a so-ring R is prime if and only if for any right ideal M and left ideal N of R. $MN \subseteq B$ implies $M \subseteq B$ or $N \subseteq B$.

Proof: Let B be a prime bi-ideal of R and $MN \subseteq B$. Suppose $M \not\subset B$. Since $MRN \subseteq MN \subseteq B$ and B is prime, $M \subseteq B$ or $N \subseteq B$. $\Rightarrow N \subseteq B$. Conversely suppose that $MN \subseteq B$ implies $M \subseteq B$ or $N \subseteq B$ for any right ideal M of R and any left ideal N of R. Let P,Q be any two bi-ideals of R such that $PRQ \subseteq B$. Now PR and RQ are right and left ideals of R. Since $(PR)(RQ) \subseteq PRQ \subseteq B$, $PR \subseteq B$ or $RQ \subseteq B$. $\Rightarrow P \subseteq B$ or $Q \subseteq B$. Hence B is prime.

3.11 Theorem. A prime bi-ideal of a so- ring R is a prime one-sided ideal of R.

Proof: Let B be a prime bi-ideal of a so-ring R. Since B is a bi- ideal of R, $(BR)(RB) \subseteq BRB \subseteq B$ where BR is a right ideal and RB a left ideal of R. By the theorem 3.10, we have that $BR \subseteq B$ or $RB \subseteq B$. Hence B is a either right or left ideal of R.

3.12 Definition. Let B be any bi-ideal of a so-ring R. Then define $L(B)$ and $H(B)$ as $\mathrm{L(B)} = \{ x \in B / Rx \subseteq B \}$ and $\mathrm{H(B)} = \{ y \in L(B) / yR \subseteq L(B) \}$.

Note that if $x \in L(B)$ and $z \in R$, then $zx \in Rx \subseteq B$ and $Rzx \subseteq RRx \subseteq Rx \subseteq B$, $L(B)$ is a left ideal of R and $L(B) \subseteq B$. Also $H(B) \subseteq L(B)$.

3.13 Theorem. If B is any bi-ideal of a so-ring R, then $H(B)$ is the (unique) largest two sided ideal of R contained in B.

Proof: Since $L(B) \subseteq B$ and $H(B) \subseteq L(B)$, we have that $H(B) \subseteq B$. Now we prove that $H(B)$ is a two sided ideal of R: Let $x \in H(B)$ and $r \in R$. Then $x \in B$ and $x \in L(B)$. $\Rightarrow Rx \subseteq B$ and $xR \subseteq L(B)$. $\Rightarrow rx \in Rx \subseteq B$ and hence $rx \in B$. Since $Rrx \subseteq Rx \subseteq B$ and $xr \in xR \subseteq L(B)$, $xr, rx \in L(B)$. Now $xrR \subseteq xR \subseteq L(B)$ and $(rx)R \subseteq RxR \subseteq RL(B) \subseteq L(B)$. Hence $xr, rx \in H(B)$. Therefore $H(B)$ is a two sided ideal of R contained in B. Now we prove that $H(B)$ is largest: Let S be any ideal of R such that $S \subseteq B$, and let u be an element of S. Then $u \in B$ and $Ru \subseteq S \subseteq B$. Hence $S \subseteq L(B)$. Also $u \in L(B)$ and $uR \subseteq S \subseteq L(B)$. $\Rightarrow u \in H(B)$ and hence $S \subseteq H(B)$. Hence the theorem.

3.14 Theorem. Let B be a prime bi-ideal of a so-ring R. Then $H(B)$ is a prime ideal of R.

Proof: Let B be a prime bi-ideal and let $XY \subseteq H(B)$ for any ideals X and Y of R. Then $XY \subseteq B$. By the theorem 3.10, $X \subseteq B$ or $Y \subseteq B$. Then by the theorem 3.13, $H(B)$ is the largest ideal contained in B. Hence $X \subseteq H(B)$ or $Y \subseteq H(B)$. Hence $H(B)$ is a prime ideal of R.

3.15 Definition. Let R be a so-ring. Then the prime radical β(R)of R is the intersection of all prime ideals of R.

3.16 Theorem. Every prime bi-ideal I of a complete so-ring R contains a minimal prime bi-ideal.

Proof: Take C = { P / P is a prime bi-ideal of R and $P \subseteq I$ }. Then $I \in$ C and hence (C ,$\subseteq$) is a non empty partial ordered set. Let $\{H_i / i \in \Delta\}$ be a descending chain of prime bi-ideals of R contained in I. Then $H = \bigcap_{i \in \Delta} H_i$ is a bi-ideal of R such that $H \subseteq I$ .To prove H is prime, let $a,b \in R$ such that $\{arb / r \in R\} \subseteq H$ and suppose $a \notin H$. Then $a \notin H_k$ for some $k \in \Delta$. Since $a \notin H_K$, $\{arb / r \in R\} \subseteq H_K$ and H_K is prime, we have $b \in H_K$. Now $\forall i \le k$, $H_k \subseteq H_i$ and hence $b \in H_i \forall i \le k, i \in \Delta$. Now $\forall i > k, H_i \subseteq H_k$ and hence $a \notin H_i$. Since $\{arb / r \in R\} \subseteq H_i$, H_i is prime and $a \notin H_i$, We have $b \in H_i \forall i > k, i \in \Delta$. $\Rightarrow b \in H_i \forall i \in \Delta$ and hence $b \in H = \bigcap_{i \in \Delta} H_i$. Hence H is a prime bi-ideal of R. Thus $H \in$ C and H is a lower bound of $\{H_i / i \in \Delta\}$ in C. Then by Zorn's lemma, C has a minimal element. Hence the theorem.

3.17 Corollary. The prime radical β(R) of a so-ring R is the intersection of all prime bi-ideals of R.

Proof: Clearly { P_i / P_i is a prime ideal of R } $\subseteq$ { B_i / B_i is a prime bi-ideal of R }.

$\Rightarrow \bigcap$ { P_i / P_i is a prime ideal of R } $\supseteq \bigcap$ { B_i / B_i is a prime bi-ideal of R }.

$\Rightarrow$ β(R) $\supseteq \bigcap$ { B_i / B_i is a prime bi-ideal of R }. We have, if B_i is a prime bi-ideal of R then $H(B_i)$ is a prime ideal of R. Then { $H(B_i)$ / $H(B_i)$ is a prime ideal of R } $\subseteq$ { P_i / P_i is a prime ideal of R }.

$\Rightarrow$β(R)= $\bigcap$ { P_i / P_i is a prime ideal of R } $\subseteq \bigcap$ { $H(B_i)$ / $H(B_i)$ is a prime ideal of R } $\subseteq \bigcap$ { B_i / B_i is a prime bi-ideal of R }. Hence β(R) = $\bigcap$ { B_i / B_i is a prime bi-ideal of R }.

4. SEMIPRIME BI-IDEALS

In this section we define semiprime bi-ideal of a so-ring *R* and characterize the prime radical interms of semiprime bi-ideals of *R*.

4.1 Definition. A proper bi-ideal I of a so-ring R is said to be *semiprime* if and only if for any bi-ideal H of R, $HRH \subseteq I$ implies $H \subseteq I$.

4.2 Example. Let $(R,\Sigma,\cdot)$ be the so-ring as in the example 3.3**.** Then for any $x \in R$, every ideal $[0,x]$ is semiprime.

Clearly every prime bi-ideal is semiprime. The following is an example of so-ring R in which a semiprime bi-ideal is not a prime bi-ideal.

4.3 Example. Let $(R,\Sigma,\cdot)$ be a so-ring as in the example 2.8**.** For the bi-ideals { 0, u }, { 0, v } and $\{0,x,y\}$ of R, { 0, u } R { 0, v } ={ 0 }$\subseteq \{0,x,y\}$. But { 0, u}$\not\subset \{0,x,y\}$ and { 0, v}$\not\subset \{0,x,y\}$. Hence $\{0,x,y\}$ is not prime. However the bi-ideal $\{0,x,y\}$ is semiprime.

4.4Theorem. If I is a bi-ideal of a complete so-ring R then the following are equivalent.

(i) I is semiprime.

(ii) $\{ara / r \in R\} \subseteq I \iff a \in I$.

Proof: (i)$\Rightarrow$(ii): Suppose I is semiprime and take $P' = \{ara / r \in R\}$.

If $a \in I$ then clearly $P' \subseteq I$. Suppose $P' \subseteq I$ and take $A = \langle a \rangle$. Let $x \in ARA$. Then $x \le \sum_i a_i r_i a_i$ for $a_i \in \langle a \rangle, r_i \in R, \forall i \in$ I. $\Rightarrow$ for any $i \in I$, $a_i \le \sum a + asa, s \in R$.

$$\Rightarrow x \le \sum_i (\sum a + asa) r_i (\sum a + asa)$$

$$= \sum_i [(\sum a) r_i (\sum a) + (\sum a) r_i (asa) + (asa) r_i (\sum a) + (asa) r_i (asa)].$$

$$= \sum_i [\sum \sum a r_i a + \sum a(r_i as)a + \sum a(sar_i)a + a(sar_i as)a].$$

$$= \sum_i \sum \sum a r_i a + \sum_i \sum a(r_i as)a + \sum_i \sum a(sar_i)a + \sum_i a(sar_i as)a$$

Since $P' \subseteq I$ and I is a bi-ideal, $x \in I$. $\Rightarrow ARA \subseteq I$. $\Rightarrow A = \langle a \rangle \subseteq I$ and hence $a \in I$.

(ii) $\Rightarrow$(i): Suppose $P' = \{ara / r \in R\} \subseteq I \Leftrightarrow a \in I$. Let A be a bi-ideal of R such that $ARA \subseteq I$ and $a \in A$. Then $\{ara / r \in R\} \subseteq ARA \subseteq I$. $\Rightarrow a \in I$ and hence $A \subseteq I$. Hence I is semiprime.

4.5 Definition. A non empty subset A of a so-ring R is a *p-system* if and only if for any $a \in A, \exists r \in R \ni ara \in A$.

Clearly every m-system is a p-system. The following is an example of a so-ring R in which a p-system is not an m-system.

4.6 Example. Let $(R, \Sigma, \cdot)$ be the so-ring as in the example 2.8. Then the sub set $\{u, v\}$ of R is a p-system. But it is not an m-system, since for $u, v \in \{u, v\}$ and for any $r \in R$, $urv = 0 \notin \{u, v\}$.

4.7 Theorem. A proper bi-ideal I of a complete so-ring R is semiprime if and only if $R \setminus I$ is a p-system.

Proof: A bi-ideal P of R is semiprime $\Leftrightarrow$ $\{ara / r \in R\} \subseteq P$ then $a \in P$ ($\because$ by theorem 4.4)

$\Leftrightarrow a \notin P$ then $\{ara / r \in R\} \not\subset P$ $\Leftrightarrow$ for any $a \in R \setminus P, \exists r \in R \ni ara \in R \setminus P$

$\Leftrightarrow R \setminus P$ is a p-system.

4.8 Theorem. Let B be a semiprime bi-ideal of a so- ring R. Then $L^2 \subseteq B$ (or $M^2 \subseteq B$) implies $L \subseteq B$ (or $M \subseteq B$) for any left ideal L (or right ideal M) of R.

Proof: Let L be a left ideal of R such that $L^2 \subseteq B$. Suppose $L \not\subset B$. Then there exists $x \in L \ni x \notin B$. $\Rightarrow xRx \subseteq LRx \subseteq LL \subseteq B$. Since B is semiprime, $x \in B$, a contradiction. Hence $L \subseteq B$. Hence the theorem.

4.9 Theorem. Let B be a semiprime bi-ideal of a so-ring R. Then $H(B)$ is a semiprime ideal of R.

Proof: Let B be a semiprime bi-ideal of R and suppose $X^2 \subseteq H(B)$ for any ideal X of R. Then $X^2 \subseteq B$. $\Rightarrow$By the above theorem, $X \subseteq B$. From the theorem 3.13, it follows that $X \subseteq H(B)$. Hence $H(B)$ is semiprime ideal of R.

4.10 Corollary. The prime radical β(R) of a so-ring R is the intersection of all the semiprime bi-ideals of R.

Proof: We have β(R) = $\bigcap$ { B_i / B_i is a prime bi-ideal of R }, we know that every prime bi-ideal is semiprime bi-ideal of R. $\Rightarrow$ { B_i / B_i is a prime bi-ideal of R } $\subseteq$ {S_i / S_i is semiprime bi-ideal of R }.$\Rightarrow \bigcap$ { B_i / B_i is a prime bi-ideal of R } $\supseteq \bigcap$ {S_i / S_i is a semiprime bi-ideal of R }.

$\Rightarrow$ β(R) $\supseteq \bigcap$ {S_i / S_i is a semiprime bi-ideal of R }. If S_i is a semiprime bi-ideal of R then $H(S_i)$ is a semiprime ideal. $\Rightarrow$ {$H(S_i)$ / $H(S_i)$ is a semiprime ideal of R } $\subseteq$ {X_i / X_i semiprime ideal of R }.$\Rightarrow$ β(R) = $\bigcap$ {X_i / X_i semiprime ideal of R } $\subseteq \bigcap$ {$H(S_i)$ / $H(S_i)$ is a semiprime ideal of R } $\subseteq \bigcap$ {S_i / S_i is semiprime bi-ideal of R }. Hence β(R) of a so-ring R is the intersection of all the semiprime bi-ideals of R.

4.11 Theorem. A partial semiring R is multiplicatively regular if and only if every bi-ideal in R is semi prime.

Proof: Let R be a multiplicatively regular partial semiring and B be any bi-ideal of R. Suppose $xRx \subseteq B$ for $x \in R$. Since R is regular, there exists $r \in R \ni x = xrx$. But $xrx \in xRx$. Hence $x \in xRx \subseteq B$ and so B is semiprime. Conversely suppose that every bi-ideal of R is semiprime. Let $r \in R$ and consider $B = rRr$. Then B is a bi-ideal of R. Hence rRr is semiprime. Since $rRr \subseteq rRr$ and rRr is semiprime, we have $r \in rRr$. $\Rightarrow \exists$ $x \in R$ such that $r = rxr$ .Hence R is a regular partial semiring.

4.12 Definition. A bi-ideal I of a so-ring R is said to be *irreducible* if and only if for any bi-ideals H and K of R, $I = H \bigcap K$ implies $I = H$ or $I = K$.

4.13 Definition. A bi-ideal I of a so-ring R is said to be *strongly irreducible* if and only if for any bi-ideals H and K of R, $H \bigcap K \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$.

In the so-ring $R = [0,1]$ as in the example 3.3, every bi-ideal $[0, x]$ is strongly irreducible. Clearly every strongly irreducible bi-ideal is irreducible. The following is an example of a so-ring R in which an irreducible bi-ideal is not a strongly irreducible bi-ideal.

4.14 Example. Let $(R, \Sigma, \cdot)$ be the so-ring as in the example 2.9**.** For the bi-ideals $\{0, a\}, \{0, b\}$ and $\{0, c\}$ of R, $\{0, b\} \bigcap \{0, c\} = \{0\} \subseteq \{0, a\}$ and $\{0, b\} \not\subset \{0, a\}, \{0, c\} \not\subset \{0, a\}$. Hence $\{0, a\}$ is not strongly irreducible. However the bi-ideal $\{0, a\}$ is irreducible.

4.15 Definition. A non empty subset A of so-ring R is said to be an *i-system* if and only if for any $a, b \in A, < a > \bigcap < b > \bigcap A \neq \phi$.

4.16 Example. Let $(R, \Sigma, \cdot)$ be the so-ring as in the example 2.8. Then the subset $\{0, u\}$ of R is an *i*-system where as the subset $\{x, y\}$ is not an *i*-system. Since $< x >= \{0, x\}, < y >= \{0, y\}$ and $< x > \bigcap < y > \bigcap A = \phi$.

4.17 Theorem. If I is a bi-ideal of a complete so-ring R then the following are equivalent :

(i) I is strongly irreducible,

(ii) if $a, b \in R$ satisfy $< a > \bigcap < b > \subseteq I$ then $a \in I$ or $b \in I$, and

(iii) $R \setminus I$ is an *i*-system.

Proof: (i)$\Rightarrow$(ii): Suppose I is strongly irreducible. Then for any $a, b \in R$ such that $< a > \bigcap < b > \subseteq I$ then $< a > \subseteq I$ or $< b > \subseteq I$. Hence $a \in I$ or $b \in I$.

(ii)$\Rightarrow$(iii): Suppose $a, b \in R$ such that $< a > \bigcap < b > \subseteq I$ imply $a \in I$ or $b \in I$. Let $a, b \in R \setminus I$. Then $< a > \bigcap < b > \not\subset I$. $\Rightarrow < a > \bigcap < b > \bigcap (R \setminus I) \neq \phi$. Hence $R \setminus I$ is an *i*-system.

(iii)$\Rightarrow$(i): Suppose $R \setminus I$ is an *i*-system. Let H, K be bi-ideals of $R \ni H \bigcap K \subseteq I$ and suppose $H \not\subset I$ and $K \not\subset I$. $\Rightarrow \exists x, y \in R \setminus I \ni x \in H$ and $y \in K$. $\Rightarrow \exists z \in < x > \bigcap < y >$ and $z \notin I$. $\Rightarrow z \in H \bigcap K$ and $z \notin I$, and hence $H \bigcap K \not\subset I$, a contradiction. Hence I is strongly irreducible.

4.18 Theorem. Let a be a non zero element of a so-ring R and let I be a bi-ideal of R not containing a. Then there exists an irreducible bi-ideal H of R containing I and not containing a.

Proof: Let C $= \{J \in Bi-ideal(R) / I \subseteq J \, \& \, a \notin J\}$. Clearly $I \in$ C. Then by Zorn's lemma , C has a maximal element. Let it be H. Now we prove that H is irreducible: Let A, B be the bi-ideals of H such that $H = A \bigcap B$ and suppose that $H \subseteq A$ and $H \subseteq B$. $\Rightarrow \exists a \in A \, \& \, a \in B$, and hence $a \in A \bigcap B = H$, a contradiction. Hence H is irreducible and hence theorem.

4.19 Theorem. Any proper bi-ideal of a so-ring R is the intersection of all irreducible bi-ideals containing it.

Proof: Let I be a proper bi-ideal of a so-ring R. $\Rightarrow 1 \notin I$. Then by the theorem 4.18**,** $\exists$ an irreducible bi-ideal J of R containing I and not containing 1. Take $I' = \bigcap \{J \in Bi-ideal(R) / J$ is irreducible and $I \subseteq J$ }. Then $I \subseteq I'$. Suppose $I \subset I'$. $\Rightarrow \exists x \in I' \ni x \notin I$. Again by the theorem 4.18, $\exists$ an irreducible bi-ideal H containing I and

$x \notin H$. Then $I' \subseteq H$ .Since $x \in I', x \in H$, a contradiction. Hence $I = I' = \bigcap\{J \in Bi\text{-}ideal(R) / J$ is irreducible and $I \subseteq J$ }.

5. Conclusion

In this paper, we introduced the notions of prime bi-ideal, m-system, semiprime bi-ideal, p-system, irreducible and strongly irreducible bi-ideals for a so-ring R . We characterized the prime radical of *R*, intersection of all prime ideals of *R* interms of prime bi-ideals and semiprime bi-ideals of *R*. Also we obtained the equivalent conditions to prime, semiprime and strongly irreducible bi-ideals of *R*.

References

[1] Acharyulu, G.V.S.: A Study of Sum-Ordered Partial Semirings, Doctoral thesis, Andhra University,1992.
[2] Amarendra Babu, V., and Srinivasa Reddy,M: Bi-ideals of Sum-Ordered Partial Semirings, International Journal of Mathematics and Computer Applications Research(IJMCAR), Vol.3, issue 2, June 2013, pp. 57-64.
[3] Arbib, M.A., Manes, E.G.: Partially Additive Categories and Flow-diagram Semantics, Journal of Algebra, 62, pp. 203-227, 1980.
[4] Le Roux, H.J.: A Note on Prime and Semiprime Bi-ideals, Kyungpook Math. J., Vol. 35, pp. 243-247,1995.
[5] Manes, E.G., and Benson, D.B.: The Inverse Semigroup of a Sum-Ordered semiring, Semigroup Forum,31, pp. 129-152, 1985.
[6] Streenstrup, M.E.: Sum-Ordered partial semirings, Doctoral thesis, Graduate school of the University of Massachusetts, Feb 1985 (Department of computer and Information Science).
[7] Srininvasa Rao, P.V.: Ideals Of Sum Ordered Partial Semirings, Intenational Journal of Computational Cognition(IJCC), Vol. 7, No. 2, pp. 59-64, June 2009.
[8] Srininvasa Rao, P.V.: Ideal Theory Of Sum-Ordered Partial Semirings, Doctoral thesis, Acharya Nagarjuna University, 2011.

AUTHOR'S BIOGRAPHY

He is working as an Assistant professor in the department of S&H in D.V.R & Dr.H.S.MIC College of Technology, Kanchikacherla, Krishna (district). He received his M. Phil., from Acharya Nagarjuna University under guidance of Dr.P.KoteswaraRao on the work "Regualr Rings ''. Now he is pursuing his Ph. D in Acharya Nagarjuna University under guidance of Dr V. Amarendra Babu in the area "Partial semirings." He has published 2 papers in international journals to his credit.

He is working as an Assistant professor in the department of Mathematics. He recieved his Ph.D degree on the work "TOPOLOGICAL A*-ALGEBRAS AND P-RINGS" under the guidance Dr.P.Koteswara Rao from Acharya Nagarjuna University in 2009. He has presented papers in various seminars and published more than 7 research papers in popular International Journals to his credit. His area of interests are Topology, Functional Analysis, Boolean Algebras, Partial semirings.

He is working as an Associate Professor in the Department of Science & Humanities, DVR & Dr. HS MIC College of Technology. He recieved the Ph. D degree on the work “Ideal Theory of Sum-ordered Partial Semirings” under the guidance Dr.N.Prabhakara Rao from Acharya Nagarjuna University in 2012. He has presented papers in various seminars and published more than 11 research papers in popular International Journals to his credit. His area of interests are Semirings, Gamma-semirings and Ordered algebras. Presently he is working on Partial Semirings, Partial Gamma-semirings & Ternary Semirings