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Characterization of Quasigroups and Loops

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Abstract: *This manuscript illustrates the significance of quasigroups when compared to general groups and subgroups. It also distinguishes the relations between groups, quasigroups, loops and equasigroups. Further various properties are verified on loops, quasigroups, equasigroups when they are compared to groups and subgroups. Various characteristics of quasigroups, loops and equasigroups were obtained in additive notation also.*

Keywords: *Quasigroups, Loops, Congruence relations, Groupoids, Semi groupoids, Groups and Sub algebras.*

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1. Introduction

Garrett Birkhoff (1942) firstly initiated the notion of lattice ordered groups. Then Bruck (1944) contributed various results in the theory of quasigroups. Zelinski (1948) described about ordered loops. The concept of non associative number theory was thoroughly studied by Evans (1957). Bruck (1963) explained about what is a loop? Various crucial properties of lattice ordered groups were established by Garrett Birkhoff in 1964 and 1967. Evans (1970) described about lattice ordered loops and quasigroups. Richard Hubert Bruck (1971) made a survey of binary systems. In the recent past Hala (1990) made a description on quasigroups and loops.

A quasigroup is a generalization of a group without associative law or identity element. Groups can be reached in another way from groupoids, namely through quasi groups. Quasigroups are simple algebraic structures whose operation has to satisfy only a single axiom the Latin square property. Hence a quasigroup is a groupoid satisfying the law a + b = c, for any two of a, b, c uniquely determines the third.

A quasigroup is a groupoid (S, +) with the property for x, y $\in$ S, there are unique elements w, z $\in$ S such that x + w = y and z + x = y. A loop is a quasigroup which has an identity element what distinguishes a loop from a group is that the former need not satisfy the associative law.

The concept of equasigroups seems more natural than that of quasigroup for two reasons. First, a non void sub set of a quasigroup Q is itself a quasi group (a " sub quasi group") if and only if it is a sub algebra of the equasi group (Q, +, \, /). And second, if θ is a congruence relation on Q, Q/θ is a quasigroup if and only if θ is a congruence relation on Q regarded as an equasigroup. That is if and only if θ has the substitution property for \ and / as well as +.

We have already the existing quasigroups and loops in multiplicative notation and here in this manuscript we make an effort to establish various properties and examples when compared with groups and groupoids in additive notation.

In this document we furnish definitions, examples and some properties of Quasigroups and Loops when compared with groups, in additive notation. In this manuscript mainly there are two topics, one is about quasigroups and the other is about loops, and the definitions, examples and properties are in additive notation. Here we provide some of the following foremost properties:

1. Every group is a quasigroup but not conversely.
2. In any quasigroup left and right cancellations laws hold.
3. Every quasigroup is an equasigroup and every equasigroup is a quasigroup.
4. A non void sub set of a quasigroup Q is itself a quasigroup (sub quasigroup) if and only if it is sub algebra of the equasigroup (Q, +.\, /).
5. Q/θ is an equasigroup. A loop is equation ally definable.
6. Any associative quasigroup is a group.
7. The class of loops is closed under direct product but not under sub algebra or epimorphic image.
8. The class of quasigroups is closed under direct product.

2. Quasigroups And Loops

2.1 Definition A quasigroup (S, +) is an algebra, where S is a non empty set, with a binary addition +, in which any two of the three terms of the equation a + b = c uniquely determine the third.

2.1.1 Example Let Z be the set of integers together with the operation usual substraction '$-$' Clearly a – c and a + b are unique solutions of a – x = c and y – b = a respectively. So (Z,$-$) is a quasigroup.

2.1.2 Note The quasigroup in the above example is not a group because it does not satisfy the associative property. In fact 2 – (3 – 1) = 2 – 2 = 0 and (2 – 3) – 1 = $-$ 1 – 1 = $-$ 2.

So every quasigroup is not a group.

2.2 Theorem Every group (G, +) is a quasigroup.

Proof: Let (G, +) be a group.
Since – a + c and a – b are the unique solutions of the equations a + x = c and y + b = a respectively.

Therefore (G, +) is a quasigroup.

Hence every group is a quasi group.

2.2.1 Note By example (2.1.1), (Z,$-$) is a quasigroup.

Write N = {1, 2…}.

Then N $\subseteq$ Z. But N is not sub algebra of (Z,$-$).

Since 2 $\in$ N, 3 $\in$ N and 2 – 3 = $-$ 1 $\notin$ N.

Therefore every quasigroup need not be a group.

2.3 Note In any quasigroup left and right cancellation laws hold.

i.e. a + x = a + y $\Longrightarrow$ x=y (left cancellation law)

$x + b = y + b \Longrightarrow x = y$ (right cancellation law)

Proof: It is clear by definition of a quasigroup.

2.4 Definition A system (S, +, \, /), where S is a nonempty set and +, \, / are binary operations on S satisfying the following identities:

(i) a+ (a\c) = c and (c/b) + b = c,
(ii) a\ (a+b) = b and (a+b)/b = a,
(iii) c/ (a\c) = a sand (c/b)\c = b, for all a, b, c in S,

is called an equasigroup.

2.4.1 Note Let (S, +) be a quasigroup. If we define a\b = −a + b, a/b = a – b then (S, +, \, /) is an equasigroup.

Proof: Let (S, +) be a quasigroup.

(i) a + (a\c) = a + (−a + c) = a − a + c = c

and (c/b) + b = c – b + b = c.

(ii) a\(a + b) = −a + a + b = b

and (a + b)/b = a + b – b = a

(iii) c/ (a\c) = c – (a\c) = c – (−a + c) = c + a – c = a

and (c/b)\c = (c – b)\c = −(c –b) + c = −c + b + c = b.

Therefore (S, +, \, /) is an equasigroup.

2.4.2 Note Every equasigroup is a quasigroup.

Proof: Let (S, +, \, /) is an equasigroup.

Therefore S satisfies the following identities.

(i) a + (a\c) = c and (c/b) + b = c.

(ii) a\ (a+b) = b and (a+b) /b = a.

(iii) c/(a\c) = a and (c/b)\c = b

The first identity state that the equations a + x = c, y + b =c have solutions and the second shows the uniqueness. Hence (S, +) is a quasigroup.

2.4.3 Note From Note (2.4.1) and Note (2.4.2), every quasigroup is equationally definable.

2.4.4 Note Equasigroup is equationally definable quasigroup.

2.4.5 Note Obviously these two definitions quasigroups and equasigroups are equivalent because x\y = z if and only if x + z = y and x/y = z if and only if z + y = x

2.5 Definition A non empty sub set H of a quasigroup S = (S, +, \, /) is called a sub quasigroup if it is closed with respect to these three operations.

That is if $x * y \in H$ for all $x, y \in H$ and $* \in \{+, \backslash, /\}$.

2.6 Lemma A non void sub set of a quasigroup Q is itself a quasigroup (sub quasigroup) if and only if it is sub algebra of the equasigroup (Q, +,\, /).

Proof: Let S be a non empty sub set of a quasigroup (Q, +).

Assume that (S, +) is a quasigroup.

So for any a, b, c $\in$ S, a + x = c, y + b = c have unique solutions in S.

That is –a + c = a\ c, c – b = c/b $\in$ S.

Clearly S is closed under +.

So S is a sub algebra of (Q, +, \, /).

Conversely assume that S is a sub algebra of the equasigroup (Q, +, \, /)

$\Rightarrow$ S is closed with respect to these three operations +, \, /.

That is (S, +, \, /) is an equasigroup.

Therefore S satisfies the following identities.

(i) a + (a\c) = c and (c/b) + b = c
(ii) a\ (a + b) = b and (a + b) / b = a.
(iii) c / (a \ c) = a and (c/ b) \ c = b

The first identity state that the equations a + x = c, y + b = c have solutions and the second shows the uniqueness.

Hence (S, +) is a quasigroup.

2.7 Definition By a congruence relation on a quasigroup (Q, +) we mean an equivalence relation θ on Q which has substitution property with respect to +, \, /.

That is $a_1 \theta b_1$, $a_2 \theta b_2$ this implies that $a_1 + a_2 \,\theta\, b_1 + b_2$, $a_1 \backslash a_2 \,\theta\, b_1 \backslash b_2$ and $a_1 / a_2 \,\theta\, b_1 / b_2$.

2.8 Lemma Let θ be congruence relation on a quasigroup (Q, +).

For any a $\in$ Q, let $\bar{a}$ denote the equivalence class containing 'a' with respect to θ.

Define the binary operations +, \, / on Q / θ by $\bar{a} + \bar{b} = \overline{a + b}$, $\bar{a} \backslash \bar{b} = \overline{a \backslash b}$, $\bar{a} / \bar{b} = \overline{a / b}$

Then Q /θ is an equasigroup.

Proof:

Part (I): First we have to prove that +, \, / are well defined.

For this, Let $\bar{a} = \bar{a}_1$ and $\bar{b} = \bar{b}_1$

So $a \equiv a_1 (\text{mod}\,\theta)$ and $b \equiv b_1 (\text{mod}\,\theta)$, since θ is a congruence relation, we have

$(a+b) \equiv (a_1+b_1)(\text{mod}\,\theta)$,

$(a\backslash b) \equiv (a_1\backslash b_1)(\text{mod}\,\theta)$,

$(a/b) \equiv (a_1/b_1)(\text{mod}\,\theta)$.

So $\overline{a + b} = \overline{a1 + b1}$, $\overline{a \backslash b} = \overline{a1 \backslash b1}$, $\overline{a/b} = \overline{a1/b1}$ .Hence +, \, / are well defined in Q/θ

Part (II): Now to prove that Q/θ is an equasigroup

(i) $\bar{a} + (\bar{a} \setminus \bar{c}) = \bar{a} + \overline{(a\setminus c)} = \overline{a + (a\setminus c)} = \bar{c}$ and

$(\bar{c}/\bar{b}) + \bar{b} = \overline{(c/b)} + \bar{b} = \overline{(c/b) + b} = \bar{c}$.

(ii) $\bar{a} \setminus (\bar{a} + \bar{b}) = \bar{a} \setminus \overline{(a+b)} = \overline{a\setminus(a+b)} = \bar{b}$ and

$(\bar{a} + \bar{b})/\bar{b} = \overline{(a+b)} / \bar{b} = \overline{(a+b)/b} = \bar{a}$.

(iii) $\bar{c} / (\bar{a} \setminus \bar{c}) = \bar{c}/\overline{(a\setminus c)} = \overline{(c/(a\setminus c))} = \bar{a}$,

$(\bar{c}/\bar{b}) \setminus \bar{c} = \overline{(c/b)} \setminus \bar{c} = \overline{((c/b)\setminus c)} = \bar{b}$

Hence Q/θ is an equasigroup**.**

2.9 Note From lemma (2.6) and lemma (2.8) equasigroup is more natural than that of quasigroup.

2.10 Definition A loop is a quasigroup (S, +) with a two sided identity 0.

That is 0+x=x+0=x for all x in S.

2.10.1Note It follows that the identity element 0 is unique and that every element of S has unique left and right inverse.

2.10.2Example 2.2 Every group is a loop, because a + x = b if and only if x = (−a) + b and y + a=b if and only if y=b+ (−a).

2.10.3Note In a loop x/x = x−x = 0 and x\x = −x + x = 0 for any x.

2.11 Theorem A loop is equationally definable.

Proof: By, Note (2.4.3) and the fact that x + 0 = 0 + x = x for any x.

Hence we have that any loop is equationally definable.

2.12 Lemma Any associative quasigroup is a group.

Proof: Let (Q, +) be an associative quasigroup.

Existence of Identity: Let a$\in$ Q.

So there exists x$\in$ Q such that a + x = a. Now we prove that x is the identity.

Let c $\in$ Q So there exists y $\in$ Q such that c = y + a.

Now c + x = (y + a) + x = y + (a + x)   (since + is associative)
= y + a = c

Also c + (x + c) = (c + x) + c = c + c.

By left cancellation law, x + c = c.

Thus x + c = c + x = c for any c $\in$ Q.
so x is additive identity. We denote this by ‘0’.

Existence of Inverse: Let a $\in$ Q.

Since Q is a quasi group there exists y$\in Q$ such that a + y = 0.

(y + a) + y = y + (a + y) = y + 0 = y = 0 + y.

By right cancellation law, y + a = 0

So y is the inverse of a.

Therefore (Q, +) is a group.

Hence any associative quasigroup is a group.

2.13 Definition Let ℓ be the class of loops. That is ℓ = {L/L is a loop}.

Let $\{L_\alpha\}_{\alpha\in\Delta}$ be a family of loops from ℓ .

Put L = $\prod_{\alpha\in\Delta} L_\alpha$

$=\{ x:\Delta \to \bigcup_{\alpha\in\Delta} L_\alpha \,/\, x(\alpha)\in L_\alpha \ \ for\ all\ \alpha\in\Delta \}.$

Then L is called direct product of all family of loops from ℓ .

2.14 Theorem The class of loops is closed under direct product but not under sub algebra or epimorphic image.

Proof: Let ℓ be the class of loops. That is ℓ = {L / L is a loop}

First we prove that ℓ is closed under direct product.

Let $\{L_\alpha\}\ \alpha\in\Delta$ is a family of loops from ℓ.

Put L= $\prod_{\alpha\in\Delta} L_\alpha$

$= \{ x:\Delta \to \bigcup_{\alpha\in\Delta} L_\alpha \,/\, x(\alpha)\in L_\alpha \ \ for\ all\ \alpha\in\Delta \}$

Define + in L by (x + y) (α)= x (α) +y(α).

Now we show that L is a loop.

Let x, y$\in L$. Let $\alpha\in\Delta$. Now x(α), y(α)$\in L_\alpha$.

Since L_α is a loop, $z_\alpha\in L_\alpha$ and $s_\alpha \in L_\alpha$ such that

$x(\alpha) + z_\alpha = y(\alpha)$ and $s_\alpha + x(\alpha) = y(\alpha)$.

Define $z:\Delta \to \bigcup_{\alpha\in\Delta} L_\alpha$ and $s:\Delta \to \bigcup_{\alpha\in\Delta} L_\alpha$ by

$z(\alpha) = z_\alpha$, $s(\alpha) = s_\alpha$.

Clearly z and s are in L.

Further x + z = y and s + x = y.

So let z^1, $s^1\in$ L be such that $x + z^1=y$ and $s^1 + x = y$.

So $x(\alpha) + z^1(\alpha) = y(\alpha)$ and $s^1(\alpha) + x(\alpha) = y(\alpha)$ for any $\alpha\in\Delta$

Since each L_α is a loop, $z^1(\alpha) = z_\alpha$ and $s^1(\alpha) = s_\alpha$ $\forall\alpha\in\Delta$

∴ $z = z^1$ and $s = s^1$ (Hence L is a loop)

Define $o:\Delta \to \bigcup_{\alpha\in\Delta} L_\alpha$ by $o(\alpha) = 0_\alpha$ (identity element of L_α)

Let $x \in L$. For any $\alpha\in\Delta$

$(x+0)(\alpha) = x(\alpha) + 0(\alpha) = x(\alpha) + 0_\alpha = x(\alpha) = 0_\alpha + x(\alpha) = 0(\alpha) + x(\alpha) = (0+x)(\alpha)$

So $x+0=x=0+x\ \forall x \in L$.

Hence O is the identity.

Hence L is a loop.

Thus ℓ is closed under direct product.

Now we prove that the class of loops is not closed under Sub algebra or epimorphic image.

It is enough if we prove that the class of loops ℓ is not closed under sub algebra.

Clearly (Z,−) is a loop.

(Z,−) is a quasi group.

Write N = {1, 2, 3…}

Then $N \subseteq Z$. But N is not sub algebra of (Z,−).

Since $2 \in N$, $3 \in N$ and $2 - 3 = -1 \notin N$.

2.15 Theorem The class of quasigroups is closed under direct product.

Proof: Same as theorem (2.14), because it is a consequence of theorem (2.14)

3. Conclusion

This research work make possible that every group is a quasigroup and every quasigroup need not be a group. A quasigroup is a generalization of a group without the associative law or identity element. It is noticed that groups can be reached in another way from groupoids, namely through quasigroups. Quasigroups are simple algebraic structures whose operation has to satisfy only a single axiom, the Latin square property. Therefore a quasigroup is a groupoid satisfying the law a + b = c, for any two of a, b, c uniquely determines the third. In any quasigroup left and right cancellation laws hold. Every equasigroup is a quasigroup. A non void sub set of a quasigroup is itself a quasigroup (sub quasigroup) if and only if it is sub algebra of the equasigroup. Further a quotient equasigroup has been derived.

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