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Lie- Theoretic of Some Generating Functions of Two Variable Laguerre Polynomials

M.B.Elkhazendar
Department of Mathematics,
Al-Azhar University – Gaza
emarawan@hotmail.com

J.M.Shenan
Department of Mathematics,
Al-Azhar University – Gaza
shenanjm@yahoo.com

T.O.Salim
Department of Mathematics,
Al-Azhar University – Gaza
trsalim@yahoo.com

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Abstract: *In the present paper, we obtain some generating functions for the Laguerre polynomials of two variable* $L_n(\mathrm{x,y})$ *with respect to* y *, by means of group Theoretical methods. The process involves the construction of a three-dimensional Lie algebra isomorphic to special linear algebra* $SL(2)$ *with help of Weisner's method by given suitable interpretations to the index* n *of the polynomials* $L_n(\mathrm{x,y})$ *.*

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1. Introduction

Dattoli and Torre [4,5] introduced and discussed a theory of two variables Laguerre polynomials. The reason of interest for this family of Laguerre polynomials is due to their intrinsic mathematical importance and to the fact that these polynomials differential equations which often appear in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings, see [13].

The two variables Laguerre polynomials (TVLP) $L_n(\mathrm{x,y})$ are specified by the series [2]

$$L_n(x,y) = \sum_{r=0}^{n} \frac{(-1)^r n! x^r y^{n-r}}{(r!)^2 (n-r)!}, \tag{1.1}$$

and the generating function for $L_n(\mathrm{x,y})$ is given by [4]

$$\sum_{n=0}^{\infty} L_n(\mathrm{x,y})\mathrm{t}^{\mathrm{n}} = \frac{1}{(1-yt)} \exp\left(\frac{-xt}{(1-yt)}\right), \quad |yt| < 1 \tag{1.2}$$

The TVLP $L_n(\mathrm{x,y})$ are linked to the polynomials $L_n(\mathrm{x})$ [1] by the relation [5, p.22 Eq. 10b]

$$L_n(\mathrm{x,\,y}) = \mathrm{y}^{\mathrm{n}} L_n\left(\frac{x}{y}\right), \; L_n(\mathrm{x,}0) = (\text{-})^{\mathrm{n}} \frac{x^n}{n!} \tag{1.3}$$

These polynomials satisfy the following differential and pure recurrence relations:

$$\frac{\partial}{\partial y}L_n(\mathrm{x},\mathrm{y})=nL_{n-1}(\mathrm{x},\mathrm{y}) \tag{1.4}$$

$$\frac{\partial^2}{\partial y^2}L_n(\mathrm{x},\mathrm{y})=n(n-1)L_{n-2}(\mathrm{x},\mathrm{y}) \tag{1.5}$$

$$L_{n+1}(\mathrm{x,y})=\frac{1}{n+1}\Big(\big((2n+1)y-x\big)L_n(\mathrm{x,y})\text{-}ny^2L_{n-1}(\mathrm{x,y})\Big) \tag{1.6}$$

$$nL_n(\mathrm{x,y})=\{(2n-1)y-x\}L_{n-1}(\mathrm{x,y})\text{-}(n-1)y^2L_{n-2}(\mathrm{x,y}) \tag{1.7}$$

Substituting $L_{n-2}(\mathrm{x,y})$ from (1.7) in (1.5) we obtain the following differential equation satisfied $L_n(\mathrm{x},\mathrm{y})$ with respect to y, we get the differential equation satisfied by $L_n(\mathrm{x},\mathrm{y})$ is

$$\left[y^2\frac{d^2}{dy^2}-\{(2n-1)y-x\}\frac{d}{dx}+n^2\right]L_n(\mathrm{x,y})=0 \tag{1.8}$$

In the present paper we utilize Weisners's [12] group-theoretic method of obtaining generating relations in the case of TVLP $L_n(\mathrm{x,y})$ with respect to y by given suitable interpretation to the index n .

It's remarked that another, more general approach to obtain Laguerre polynomials of many variables (and other polynomials as well, such as Hermite, Charlier) using a chaos decomposition on spaces of Fock type can be found in [6] and the reference there in. However, the direct and appropriate for obtaining generating relations.

The process involves the construction of a three-dimensional Lie algebra isomporphic to the special linear algebra $sl(2)$, lie algebra of $SL(2)$([8], p.7).The 2×2 complex special linear group $SL(2)$ is the abstract matrix group of all 2×2 non-singular matrices,

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a,b,c,d\in C$$

Such that $\det g=1$, $SL(2)$ is a three dimensional local Lie group. The elements

$$J^+=\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\quad J^-=\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\quad J^3=\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

satisfying the commutation relations

$$[J^3,J^+]=J^+,\quad [J^3,J^-]=-J^-,\quad [J^3,J^-]=2J^3$$

from a basis for $sl(2)$.

2. GROUP- THEORETIC METHOD

By replacing d/dx by $\partial/\partial x$ and n by $z\dfrac{\partial}{\partial z}$ in Equation (1.8) we construct a partial differential equation:

$$\left(y^2\frac{\partial^2}{\partial y^2}-\left[(2z\frac{\partial}{\partial z}-1)y-x\right]\frac{\partial}{\partial y}+z^2\frac{\partial^2}{\partial z^2}\right)f(x,y;z)=0 \tag{2.1}$$

Thus, $f(x,y;z)=L_n(\text{x,y})z^n$ is a solution of equation (2.1) since $L_n(\text{x,y})$ is a solution of equation (1.8).

First we consider the following first –order linear partial differential operators

$$\left.\begin{aligned}&J^3=z\frac{\partial}{\partial z}+\frac{1}{2};\\&J^+=y^2z\frac{\partial}{\partial y}-2yz^2\frac{\partial}{\partial z}+(x-y)z\\&J^-=\frac{1}{z}\frac{\partial}{\partial x}\end{aligned}\right\} \tag{2.2}$$

such that

$$J^3\left[L_n(\text{x,y})z^n\right]=(n+\frac{1}{2})L_n(\text{x,y})z^n.$$

$$J^+\left[L_n(\text{x,y})z^n\right]=-(n+\frac{1}{2})L_{n+1}(\text{x,y})z^{n+1}.$$

$$J^-\left[L_n(\text{x,y})z^n\right]=nL_{n-1}(\text{x,y})z^{n-1}.$$

These operators satisfy the commutation relations

$$[J^3,J^+]=J^+,\quad [J^3,J^-]=-J^-,\quad [J^+,J^-]=-2J^3 \tag{2.3}$$

where $[A,B]=AB-BA$.

The above commutation relation show that the set of J-operators $\{J^3,J^+,J^-\}$ generates a three – dimensional Lie algebra isomorphic to $sl(2)$, the Lie Algebra of $SL(2)$.

In terms of the J-operators, we introduce the Casimir operator [8, p.32]

$$\begin{aligned}C&=J^+J^-+J^3J^3-2J^3\\&=\left(y^2\frac{\partial}{\partial y^2}-2zy\frac{\partial}{\partial z}+y\frac{\partial}{\partial y}+x\frac{\partial}{\partial y}+z^2\frac{\partial}{\partial z^2}\right)-1\end{aligned} \tag{2.4}$$

We can verify that C commutes with J^3,J^+ and J^-, that is

$$[C,J^3]=[C,J^+]=[C,J^-]=0 \tag{2.5}$$

Equation (2.4) enables us to write Equation (2.1) as

$$Cf(x,y;z)=(-1)f(x,y;z) \tag{2.6}$$

Now we proceed the commute the multiplier representation $[T(g)f](x,y;z),\ g\in SL(2),$ induced by the J-operators (2.2).

A simple computation using [11. P.320, Theorem 7] and (2.2) gives

$$T\left[\left(\exp(b'J^{+})\right)f\right](x,y,z)=$$

$$(1-b'yz)^{-1}\exp\left(\frac{-b'xz}{1-b'yz}\right)\times f\left(x,\frac{y}{1-b'yz};\frac{z}{1+zb'yz}\right),\ |b'yz|<1,$$

$$T\left[\left(\exp(c'J^{-})\right)f\right](x,y,z)=f\left(x,\frac{y}{1-\frac{c'}{yz}};z\right),\ \left|\frac{c'}{yz}\right|<1, \tag{2.7}$$

$$T\left[\left(\exp(\tau'J^{3})\right)f\right](x,y,z)=\exp\left(\frac{\tau'}{2}\right)f\left(x,y,z\exp(\tau')\right),$$

Defined for $|b'|,|c'|$ and $|\tau'|$ sufficiently small. If $g\in SL(2)$ and $d\neq 0$, it is a straightforward computation to show that $g=\exp(b'J^{+})\exp(c'J^{-})\exp(\tau'J^{3})$, where $b'=\frac{-b}{d}, c'=-cd, \exp\left(\frac{\tau'}{2}\right)=\frac{1}{d}=a,$ and $ad-bc=1$.

Hence the operator $T(g)$ is given by

$$[T(g)f](x,y,z)=(d+byz)^{-1}\exp\left(\frac{bxz}{d+byz}\right)\times f\left(x,\frac{y^{2}z}{(d+byz)(x+ayz)},\frac{az}{d-2byz}\right),$$

$$\max\left(\left|\frac{byz}{d}\right|,\left|\frac{c}{ayz}\right|\right)<1,\quad |\arg(d)|<\pi \tag{2.8}$$

3. GENERATING RELATIONS

To accomplish our task of obtaining generating relations, we search for the function $f(x,y,z)$ which satisfies Equation (2.6). Consider the case when $f(x,y,z)$ is a common eigen function of C and J^{3}, that is, let $f(x,y,z)$ be a solution of the simultaneous equations

$$Cf(x,y;z)=(-1)f(x,y;z)$$

$$J^{3}f(x,y;z)=\left(n+\frac{1}{2}\right)f(x,y;z) \tag{3.1}$$

Which can be written as

$$\left(y^{2}\frac{\partial^{2}}{\partial y^{2}}+\left[(2z\frac{\partial}{\partial z}-1)y-x\right]\frac{\partial}{\partial y}+z^{2}\frac{\partial^{2}}{\partial z^{2}}\right)f(x,y;z)=0$$

$$\left(z\frac{\partial}{\partial z}-n\right)f(x,y;z)=0 \tag{3.2}$$

Equation (3.2) yields $f(x,y;z)=L_{n}(x,y)z^{n}$, so that we have

$$\left[(T(g))f\right](x,y,z)=(d+byz)^{-1}\exp\left(\frac{bxz}{d+byz}\right)\left(\frac{az}{d-2ybz}\right)^{n}\times L_{n}\left(x,\frac{y^{2}z}{(d+byz)(c+ayz)}\right), \tag{3.3}$$

satisfying the relation

$C\left[T(g)\right]f(x,y;z)=(-1)\left[T(g)f\right](x,y;z)$

If n is a non-negative integer, (3.3) has an expression of the form

$$\left[T(g)f\right](x,y;z)=\sum_{k=0}^{\infty}A_{kn}(g)L_k(x,y)z^k$$

which simplifies to the identity

$$(d+byz)^{-1}\left(\frac{az}{d-2ybz}\right)^n\exp\left(\frac{bxz}{d+byz}\right)L_n\left(x,\frac{y^2z}{(d+byz)(c+ayz)}\right)=\sum_{k=0}^{\infty}A_{kn}(g)L_k(x,y)z^k \tag{3.4}$$

To determine $A_{kn}(g)$, we set $x=0$ and $y=1$ in equation (3.4), and thus we have

$$A_{kn}(g)=\sum_{r=0}^{\infty}\frac{(-n)_r(n-r+1)_{k-n+r}}{(k-n+r)!r!}(-b)^{k-n+r}(-c)^r d^{-n-k-1}\quad ,k,n\geq 0 \tag{3.5}$$

Substituting (3.5) into equation (3.4) and simplifying, we obtain the generating relation

$$(ad)^n\left(a+\frac{byz}{d}\right)^{-n-1}\left(\frac{z}{d-2ybz}\right)^n\exp\left(\frac{bxz}{d+byz}\right)L_n\left(x,\frac{y^2z}{(d+byz)(c+ayz)}\right)=\sum_{k=0}^{\infty}\sum_{r=0}^{n}\frac{(-n)_r(n-r+1)_k}{k!r!}\left(-\frac{bz}{d}\right)^k\left(-\frac{cd}{z}\right)^r L_{n+k-r}(x,y), \tag{3.6}$$

We consider some special cases of (3.6).

CASE I:

Taking $a=c=d=1,b=0$ and replacing z by $-\frac{1}{t}$ in (3.6), we obtain

$$\left(\frac{-1}{t}\right)^n L_n\left(x,\frac{y}{1-t/y}\right)=\sum_{r=0}^{n}\frac{(-n)_r}{r!}L_{n-r}(x,y)t^r, \tag{3.7}$$

CASE II:

Taking $a=b=d=1,c=0$ and replacing z by $-t$ in (3.6), we obtain

$$\left(1-yt\right)^{-n-1}\left(\frac{-t}{1+2yt}\right)^n\exp\left(\frac{-xt}{1-yt}\right)L_n\left(x,\frac{y}{1-yt}\right)=\sum_{k=0}^{n}\frac{(n+1)_k}{k!}L_{n+k}(x,y)t^k, \tag{3.8}$$

CASE III:

Taking $a=d=1$ and $bc\neq 0$ without any loss of generality we can choose $bz=-t_1$ and $c/z=-t_2$ in (3.6), we obtain

$$(1-yt_1)^{-n-1}\left(\frac{z}{1+2yt_1}\right)^n \exp\left(\frac{-xt_1}{1-yt_1}\right)L_n\left(x,\frac{y}{(1-yt_1)(1-t_2/y)}\right)$$
$$=\sum_{k=0}^{n}\sum_{r=0}^{\infty}\frac{(-n)_r(n-r+1)_k}{k!r!}t_1^k t_2^r L_{n+k-r}(x,y), \tag{3.9}$$

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AUTHOR'S BIOGRAPHY

Marawan Baker Elkhazendar is working as Associate professor of Mathematics in the Department of Mathematics, Al-Azhar university – Gaza, Palestine. He obtained his Ph.D from Jodhpur university, India, 2003. His research is focused on the topics: Lie Algebra and special functions. He has many research published papers in his area of research. He attended and organized several conferences in Mathematics and science. His teaching experience is more than 20 years where he taught several undergraduate and graduate courses like: Calculus ,Linear Algebra , Abstract Algebra , Principles of Mathematics, Real Analysis ,Discrete Mathematics , Advanced Linear Algebra, Advanced Modern Algebra ,Number Theory, Mathematical Analysis, Vector Analysis.

Jamal Mohammed Ali Shenan has been working as Associate professor of Mathematics in the Department of Mathematics, Al-Azhar university –Gaza, Palestine. He obtained Ph.D from Jodhpur university, India, *2002* . His research is focused on the topics: fractional calculus, special functions, analytic functions. He has many research published papers in his area of research. He attended and organized several conferences in Mathematics and science. His teaching experience is more than 15 years where He taught several undergraduate and graduate courses like: calculus, Probability theory, Mathematical statistics, Complex analysis. Elementary statistics. Principal of mathematics, Functional Analysis (B. Sc.), Functional Analysis (M. Sc.), Discrete mathematics.

Tariq Omar Salim is a professor of Mathematics in the Department of Mathematics, Al-Azhar university –Gaza, Palestine. He had his Ph.D from University of Rajasthan, Jaipur India in 1999. His research is focused on the topics: fractional calculus, special functions, analytic functions. He has more than 40 research published papers in his area of research. He attended and organized several conferences in Mathematics and science. His teaching experience is more than 25 years where He taught several undergraduate and graduate courses like: calculus, real analysis, complex analysis, special functions, integral equations. Also He supervised several graduate students.