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A New Aspect for the Two-Dimensional Quantum Measurement Theories

Koji Nagata[1], Tadao Nakamura[2]

[1]Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon, Korea
[2]Department of Information and Computer Science, Keio University, Hiyoshi, Kohoku-ku, Yokohama, Japan
ko_mi_na@yahoo.co.jp, nakamura@pipelining.jp

Recently, a new measurement theory based on the truth values is proposed [38]. The results of measurements are either 0 or 1. The measurement theory accepts a hidden variables model for a single Pauli observable. Therefore we can introduce a classical probability space for the measurement theory in this case. On the other hand, we discuss the fact that the projective measurement theory (the results of measurements are either +1 or -1) says the Bell, Kochen, and Specker (BKS) paradox for the single Pauli observable. To justify our assertion, we present the BKS theorem in almost all the two-dimensional states, by using the projective measurement theory. As an example, we present the BKS theorem in the two-dimensional white noise state, by using the projective measurement theory. Our discussion provides new insight to formulate quantum measurement theory, by using the measurement theory based on the truth values.

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1. INTRODUCTION

The projective measurement theory (cf. [1-6]) gives at times remarkably accurate numerical predictions. From the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [7], a hidden-variables interpretation of quantum mechanics has been an attractive topic of research [3, 4]. One is the Bell-EPR theorem [8]. Another is the no-hidden-variables theorem of Kochen and Specker (the KS theorem) [9]. Greenberger, Horne, and Zeilinger discover [10, 11] the so-called GHZ theorem for four-partite GHZ state. And, the Bell-KS theorem becomes very simple form (see also Refs. [12-16]). The Leggett-type nonlocal hidden-variable theory [17] is experimentally investigated [18-20]. The experiments report that quantum mechanics does not accept the Leggett-type nonlocal hidden-variable theory. These experiments are performed in four-dimensional space (two parties) in order to study nonlocality of the hiddenvariable theory. However there are debates for the conclusions of the experiments. See Refs. [21-23]. For the applications of quantum mechanics, an implementation of a quantum algorithm to solve Deutsch's problem [24-26] on a nuclear magnetic resonance quantum computer is reported firstly [27]. An implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [28]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira et al. implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [29]. Single-photon Bell states are prepared and measured [30]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [31]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [32]. In 1993, the Bernstein-Vazirani algorithm was reported [33]. It can be considered as an extended Deutsch-Jozsa algorithm. In 1994, Simon's algorithm was reported [34]. An implementation of a quantum algorithm to solve the Bernstein-Vazirani parity problem without an entanglement on an ensemble quantum computer is reported [35]. A fiber-optics implementation of the Deutsch-Jozsa and Bernstein-Vazirani quantum algorithms with three qubits is discussed [36]. A quantum algorithm for approximating the influences of Boolean functions and its applications is recently reported [37].

Recently, a new measurement theory based on the truth values is proposed [38]. The results of measurements are either 0 or 1. We do not know the complete differences between the project measurement theory (The results of measurements are either +1 or -1) and the measurement theory based on the truth values. Here we investigate one of the differences of them.

The new measurement theory accepts a hidden variables model for a single Pauli observable. Therefore we can introduce a classical probability space for the measurement theory in this case. On the other hand, we discuss the fact that the projective measurement theory (the results of measurements are either +1 or -1) says the Bell, Kochen, and Specker (BKS) paradox for the single Pauli observable. To justify our assertion, we present the BKS theorem in almost all the two-dimensional states, by using the projective measurement theory. As an example, we present the BKS theorem in a two-dimensional white noise state, by using the projective measurement theory. Our discussion provides new insight to formulate quantum measurement theory, by using the measurement theory based on the truth values.

2. The Measurement Theory Based on the Truth Values Meets A Hidden Variables Model of A Single Spin Observable

We discuss the new measurement theory meets a hidden variables model of a single spin observable. Assumea spin-1/2 state ρ . Let σ_x be a single Pauli observable.We have a quantum expected value as

$$Tr[\rho\sigma_x] \tag{1}$$

We derive a necessary condition for the quantum expectedvalue for the system in a spin-1/2 state given in(1). We have

$$0 \le (Tr[\rho\sigma_x])^2 \le 1 \tag{2}$$

It is worth noting here that we have $(Tr[\rho\sigma_x])^2 = 1$ if ρ isthe pure state lying in the x-direction. Hence we derivethe following proposition concerning quantum mechanicswhen the system is in the state lying in the x-direction

$$(Tr[\rho\sigma_x])^2_{\max} = 1 \tag{3}$$

$(Tr[\rho\sigma_x])^2_{\max}$ is the maximal possible value of the product.It is worth noting here that we have $(Tr[\rho\sigma_x])^2 = 0$ when the system is in the pure state lying in the z-direction. Thus we have

$$(Tr[\rho\sigma_x])^2_{\min} = 0 \tag{4}$$

$(Tr[\rho\sigma_x])^2_{\min}$ is the minimal possible value of the product. In short, we have

$$(Tr[\rho\sigma_x])^2_{\min} = 0 \text{ and } (Tr[\rho\sigma_x])^2_{\max} = 1 \tag{5}$$

In what follows, we derive the above proposition (5) assuming the following form:

$$Tr[\rho\sigma_x] = \int d\lambda\rho(\lambda) f(\sigma_x,\lambda) \tag{6}$$

Where λ denotes some hidden variable and $f(\sigma_x,\lambda)$ is the hidden result of measurements of the Pauli observable σ_x .We assume that the values of $f(\sigma_x,\lambda)$ are either 1 or 0(in ħ/2 unit).

Let us assume the hidden variables theory of the singlespin observable based on the new measurement theory. Inthis case, the quantum expected value in (1), which is theaverage of the hidden results of the new measurements,is given by

$$Tr[\rho\sigma_x] = \int d\lambda\rho(\lambda) f(\sigma_x,\lambda) \tag{7}$$

The possible values of the hidden result $f(\sigma_x,\lambda)$ are either1 or 0 (in ħ/2 unit). The same expected value isgiven by

$$Tr[\rho\sigma_x] = \int d\lambda'\rho(\lambda') f(\sigma_x,\lambda') \tag{8}$$

Because we only change the notation as $\lambda \to \lambda'$. Ofcourse, the possible values of the hidden result $f(\sigma_x, \lambda')$ are either 1 or 0 (in ħ/2 unit). By using these facts, we derive a necessary condition for the expected value forthe system in the spin-1/2 state lying in the x-direction.We derive the possible values of the product $(Tr[\rho\sigma_x])^2$.We have

$$(Tr[\rho\sigma_x])^2$$

$$= \int d\lambda\rho(\lambda) f(\sigma_x, \lambda) \times \int d\lambda'\rho(\lambda') f(\sigma_x, \lambda')$$

$$= \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda') f(\sigma_x, \lambda) f(\sigma_x, \lambda')$$

$$\leq \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda') \left| f(\sigma_x, \lambda) f(\sigma_x, \lambda') \right|$$

$$= \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda') = 1 \tag{9}$$

Clearly, the above inequality can have the upper limitsince the following casesare possible:

$$\left\| \{\lambda \mid f(\sigma_x, \lambda) = 1\} \right\| = \left\| \{\lambda' \mid f(\sigma_x, \lambda') = 1\} \right\| \tag{10}$$

and

$$\left\| \{\lambda \mid f(\sigma_x, \lambda) = 0\} \right\| = \left\| \{\lambda' \mid f(\sigma_x, \lambda') = 0\} \right\| \tag{11}$$

Thus we derive a proposition concerning the hidden variables theory based on the new measurement theory (ina spin-1/2 system), that is, $(Tr[\rho\sigma_x])^2 \leq 1$. Hence we derive the following proposition concerning the hidden variables theory:

$$(Tr[\rho\sigma_x])^2_{\max} = 1 \tag{12}$$

We derive another necessary condition for the expected value for the system in the pure spin-1/2 state lying inthe z-direction. We have

$$(Tr[\rho\sigma_x])^2$$

$$= \int d\lambda\rho(\lambda) f(\sigma_x, \lambda) \times \int d\lambda'\rho(\lambda') f(\sigma_x, \lambda')$$

$$= \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda') f(\sigma_x, \lambda) f(\sigma_x, \lambda')$$

$$\geq \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')(0)$$

$$= (0)(\int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')) = 0 \tag{13}$$

Clearly, the above inequality can have the lower limit since the following case is possible:

$$\left\| \{\lambda \mid f(\sigma_x, \lambda) = 1\} \right\| = \left\| \{\lambda' \mid f(\sigma_x, \lambda') = 0\} \right\| \tag{14}$$

and

$$\left\| \{\lambda \mid f(\sigma_x, \lambda) = 0\} \right\| = \left\| \{\lambda' \mid f(\sigma_x, \lambda') = 1\} \right\| \tag{15}$$

Thus we derive a proposition concerning the hidden variables theory based on the new measurement theory (ina spin-1/2 system), that is, $(Tr[\rho\sigma_x])^2 \geq 0$. Hence we derive the following proposition concerning the hidden variables theory

$$(Tr[\rho\sigma_x])^2_{\min} = 0 \tag{16}$$

Thus from (12) and (16) we have

$$(Tr[\rho\sigma_x])^2_{min} = 0 \text{ and } (Tr[\rho\sigma_x])^2_{max} = 1 \tag{17}$$

Clearly, we can assign the truth value "1" for the two propositions (5) (concerning quantum mechanics) and(17) (concerning the hidden variables theory based on the new measurement theory), simultaneously. Therefore,the new measurement theory meets the existence of the hidden variables theory of the single spin observable.

3. The Projective Measurement Theory Does Not Meet A Hidden Variables Model of A Single Spin Observable

In what follows, we cannot derive the proposition (5) assuming the following form:

$$Tr[\rho\sigma_x] = \int d\lambda\rho(\lambda) f(\sigma_x,\lambda) \tag{18}$$

Where λ denotes some hidden variable and $f(\sigma_x,\lambda)$ is the hidden result of measurements of the Pauli observable σ_x.We assume that the values of $f(\sigma_x,\lambda)$ are either +1 or -1 (in ħ/2 unit).

Let us assume a hidden variables model based on the projective measurement theory of the single spin observable. In this case, the quantum expected value in (1), which is the average of the hidden results of the projective measurements, is given by

$$Tr[\rho\sigma_x] = \int d\lambda\rho(\lambda) f(\sigma_x,\lambda) \tag{19}$$

The possible values of the hidden result $f(\sigma_x,\lambda)$ are either+1 or -1 (in ħ/2 unit). The same expected value isgiven by

$$Tr[\rho\sigma_x] = \int d\lambda'\rho(\lambda') f(\sigma_x,\lambda') \tag{20}$$

because we only change the notation as $\lambda \rightarrow \lambda'$. Ofcourse, the possible values of the hidden result $f(\sigma_x,\lambda')$ are either +1 or -1 (in ħ/2 unit). By using these facts, we derive a necessary condition for the expected value forthe system in the spin-1/2 state lying in the x-direction.We derive the possible values of the product $(Tr[\rho\sigma_x])^2$.

We have

$$\begin{aligned}
&(Tr[\rho\sigma_x])^2 \\
&= \int d\lambda\rho(\lambda) f(\sigma_x,\lambda) \times \int d\lambda'\rho(\lambda') f(\sigma_x,\lambda') \\
&= \int d\lambda\rho(\lambda)\cdot\int d\lambda'\rho(\lambda') f(\sigma_x,\lambda) f(\sigma_x,\lambda') \\
&\le \int d\lambda\rho(\lambda)\cdot\int d\lambda'\rho(\lambda')\left|f(\sigma_x,\lambda) f(\sigma_x,\lambda')\right| \\
&= \int d\lambda\rho(\lambda)\cdot\int d\lambda'\rho(\lambda') = 1
\end{aligned} \tag{21}$$

Clearly, the above inequality can have the upper limit since the following cases are possible:

$$\left\|\{\lambda \mid f(\sigma_x,\lambda) = 1\}\right\| = \left\|\{\lambda' \mid f(\sigma_x,\lambda') = 1\}\right\| \tag{22}$$

and

$$\left\|\{\lambda \mid f(\sigma_x,\lambda) = 0\}\right\| = \left\|\{\lambda' \mid f(\sigma_x,\lambda') = 0\}\right\| \tag{23}$$

Thus we derive a proposition concerning the hidden variables theory based on the projective measurement theory (in a spin-1/2 system), that is, $(Tr[\rho\sigma_x])^2 \le 1$. Hence we derive the following proposition concerning the hidden variables theory

$$(Tr[\rho\sigma_x])^2_{max} = 1 \tag{24}$$

We derive another necessary condition for the expected value for the system in the pure spin-1/2 state lying inthe z-direction.

We introduce an assumption that Sum rule and Product rule commute with each other [40]. We do not pursue the details of the assumption. To pursue the details is an interesting point. It is suitable to the next step of researches. We have

$$(Tr[\rho\sigma_x])^2$$

$$=\int d\lambda\rho(\lambda)f(\sigma_x,\lambda)\times\int d\lambda'\rho(\lambda')f(\sigma_x,\lambda')$$

$$=\int d\lambda\rho(\lambda)\cdot\int d\lambda'\rho(\lambda')f(\sigma_x,\lambda)f(\sigma_x,\lambda')$$

$$\geq\int d\lambda\rho(\lambda)\cdot\int d\lambda'\rho(\lambda')(-1)$$

$$=(-1)(\int d\lambda\rho(\lambda)\cdot\int d\lambda'\rho(\lambda'))=-1 \quad (25)$$

Clearly, the above inequality can have the lower limit since the following cases are possible:

$$\left\|\{\lambda\,|\,f(\sigma_x,\lambda)=1\}\right\|=\left\|\{\lambda'\,|\,f(\sigma_x,\lambda')=-1\}\right\| \quad (26)$$

and

$$\left\|\{\lambda\,|\,f(\sigma_x,\lambda)=-1\}\right\|=\left\|\{\lambda'\,|\,f(\sigma_x,\lambda')=1\}\right\| \quad (27)$$

Thus we derive a proposition concerning the hidden variablestheory based on the projective measurement theory (in a spin-1/2 system), that is, $(Tr[\rho\sigma_x])^2\geq -1$. Hence we derive the following proposition concerning the hidden variables theory

$$(Tr[\rho\sigma_x])^2_{\min}=-1 \quad (28)$$

Thus from (24) and (28) we have

$$(Tr[\rho\sigma_x])^2_{\min}=-1 \text{ and } (Tr[\rho\sigma_x])^2_{\max}=1 \quad (29)$$

Clearly, we cannot assign the truth value "1" for two propositions (5) (concerning quantum mechanics) and (29) (concerning the hidden variables theory based onthe projective measurement theory),simultaneously. Infact, we are in the BKS contradiction. Therefore, the projective measurement theory does not meet the existenceof the hidden variables theory of the single spin observable.

4. THE BKS THEOREM IN ALMOST ALL THE TWO-DIMENSIONAL STATES

In this section, we present the BKS theorem in almost all the two-dimensional states.

4.1. Wave Function Analysis

Let σ_z be a single Pauli observable. Here,

$$\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (30)$$

We assume that a source of a spin-carrying particle emitssome of themselves in a state ρ . ρ is not the eigenvector of σ_z . Thus,

$$\rho\neq\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad (31)$$

and

$$\rho\neq\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \quad (32)$$

We consider a quantum expected value ($Tr[\rho\sigma_z]$). If we consider only a wave function analysis, the possible valuesof the square of the quantum expected value are

$$(\mathrm{Tr}[\rho\sigma_z])^2 = Z, \ (0 \leq Z < 1) \tag{33}$$

We define $\|E_{QM}\|^2$ as

$$\|E_{QM}\|^2 = (\mathrm{Tr}[\rho\sigma_z])^2 \tag{34}$$

We have

$$\|E_{QM}\|^2 \leq Z \tag{35}$$

Thus,

$$\|E_{QM}\|^2_{\max} = Z \tag{36}$$

Where $\|E_{QM}\|^2_{\max}$ is the maximal possible value of the product. Hence we have

$$\|E_{QM}\|^2_{\max} = Z, \ (0 \leq Z < 1) \tag{37}$$

4.2. Realistic Theory

A mean value E satisfies a realistic theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_z)}{m} \tag{38}$$

Where l denotes a notation and r is the result of the measurements of the Pauli observable σ_z. We assume the values of r are either +1 or -1 (in $\hbar/2$ unit). Assume the quantum mean value with the system in the state admits the realistic theory. One has the following proposition concerning the realistic theory

$$Tr[\rho\sigma_z](m) = \frac{\sum_{l=1}^{m} r_l(\sigma_z)}{m} \tag{39}$$

We can assume the following by Strong Law of Large Numbers [39],

$$Tr[\rho\sigma_z](+\infty) = Tr[\rho\sigma_z] \tag{40}$$

We define $\|E_{QM}\|^2(m)$ as

$$\|E_{QM}\|^2(m) = (Tr[\rho\sigma_x](m))^2 \tag{41}$$

We can assume the following by Strong Law of Large Numbers,

$$\|E_{QM}\|^2(+\infty) = \|E_{QM}\|^2 = (Tr[\rho\sigma_x])^2 \tag{42}$$

In what follows, we show that we cannot accept the relation (39) concerning the realistic theory. Assume the proposition (39) is true. By changing the notation l into l', we have the same quantum mean value as follows

$$Tr[\rho\sigma_z](m) = \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_z)}{m} \tag{43}$$

We introduce the assumption that Sum rule and Product rule commute with each other [40]. We have the following

$$\left\| E_{QM} \right\|^2 (m) = \frac{\sum_{l=1}^{m} r_l(\sigma_z)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_z)}{m}$$

$$\leq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} \left| r_l(\sigma_z) r_{l'}(\sigma_z) \right|$$

$$= \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} = 1 \tag{44}$$

Clearly, the above inequality can have the upper limit since the following cases are possible:

$$\left\| \{ l \left| l \in N \wedge r_l(\sigma_z) = 1 \} \right\| = \left\| \{ l' \right| l' \in N \wedge r_{l'}(\sigma_z) = 1 \} \right\| \tag{45}$$

and

$$\left\| \{ l \left| l \in N \wedge r_l(\sigma_z) = -1 \} \right\| = \left\| \{ l' \right| l' \in N \wedge r_{l'}(\sigma_z) = -1 \} \right\| \tag{46}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the realistic theory is true (in a spin-1/2 system), that is

$$\left\| E_{QM} \right\|^2 (m) \leq 1 \tag{47}$$

From Strong Law of Large Numbers, we have

$$\left\| E_{QM} \right\|^2 \leq 1 \tag{48}$$

Hence we derive the following proposition concerning the realistic theory

$$\left\| E_{QM} \right\|^2_{\max} = 1 \tag{49}$$

We cannot accept the two relations (37) (concerning the wave function analysis) and (49) (concerning the realistic theory), simultaneously. Hence we are in the BKS contradiction.

The realistic theory does not meet the wave function analysis and cannot simulate almost all the two dimensional states. The exceptions are the eigenstates of the measured spin observable.

5. The BKS Theorem in a Two-Dimensional White Noise State

In this section, we present the BKS theorem in the two-dimensional white noise state.

5.1. Wave Function Analysis

Let σ_x be a single Pauli observable. Here,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{50}$$

We assume that a source of a spin-carrying particle emits some of themselves in a state V_{noise}. Here,

$$V_{noise} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{51}$$

We consider a quantum expected value $Tr[V_{noise}\sigma_x]$. If we consider only a wave function analysis, the possible value of the square of the quantum expected value is

$$(Tr[V_{noise}\sigma_x])^2 = 0 \tag{52}$$

We define $\left\|E_{QM}\right\|^2$ as

$$\left\|E_{QM}\right\|^2 = (Tr[V_{noise}\sigma_x])^2 \tag{53}$$

$\left\|E_{QM}\right\|^2_{\max}$ and $\left\|E_{QM}\right\|^2_{\min}$ are the maximal and minimal possible values of the product, respectively. We have

$$\left\|E_{QM}\right\|^2 \leq 0 \tag{54}$$

Thus,

$$\left\|E_{QM}\right\|^2_{\max} = 0 \tag{55}$$

We have

$$\left\|E_{QM}\right\|^2 \geq 0 \tag{56}$$

Thus,

$$\left\|E_{QM}\right\|^2_{\min} = 0 \tag{57}$$

Hence we have

$$\left\|E_{QM}\right\|^2_{\min} = 0 \text{ and} \left\|E_{QM}\right\|^2_{\max} = 0 \tag{58}$$

5.2. Realistic Theory

A mean value E satisfies a realistic theory if it can bewritten as

$$E = \frac{\sum_{l=1}^{m} r_1(\sigma_x)}{m} \tag{59}$$

Where l denotes a notation and *r* is the result of the measurements of the Pauli observable σ_x. We assumethe values of *r* are either 1 or -1 (in $\hbar/2$ unit). Assume the quantum mean value with the system in the two dimensional white noise state admits the realistic theory. One has the following proposition concerning the realistic theory

$$Tr[V_{noise}\sigma_x](m) = \frac{\sum_{l=1}^{m} r_1(\sigma_x)}{m} \tag{60}$$

We can assume the following by Strong Law of Large Numbers [39],

$$Tr[V_{noise}\sigma_x](+\infty) = Tr[V_{noise}\sigma_x] \tag{61}$$

We define $\left\|E_{QM}\right\|^2(m)$ as

$$\left\|E_{QM}\right\|^2(m) = (Tr[V_{noise}\sigma_x](m))^2 \tag{62}$$

We can assume the following by Strong Law of Large Numbers,

$$\left\|E_{QM}\right\|^2(+\infty) = \left\|E_{QM}\right\|^2 = (Tr[V_{noise}\sigma_x])^2 \tag{63}$$

In what follows, we show that we cannot accept the relation (60) concerning the realistic theory. Assume the proposition (60) is true. By changing the notation l into l', we have the same quantum mean value as follows

$$Tr[V_{noise}\sigma_x](m) = \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m} \tag{64}$$

We introduce the assumption that Sum rule and Product rule commute with each other [40]. We have the following

$$\left\|E_{QM}\right\|^2(m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m}$$

$$\leq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} \left| r_l(\sigma_x) r_{l'}(\sigma_x) \right|$$

$$= \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} = 1 \tag{65}$$

Clearly, the above inequality can have the upper limit since the following cases are possible:

$$\left\|\{l \mid l \in N \wedge r_l(\sigma_x) = 1\}\right\| = \left\|\{l' \mid l' \in N \wedge r_{l'}(\sigma_x) = 1\}\right\| \tag{66}$$

and

$$\left\|\{l \mid l \in N \wedge r_l(\sigma_x) = -1\}\right\| = \left\|\{l' \mid l' \in N \wedge r_{l'}(\sigma_x) = -1\}\right\| \tag{67}$$

And we have the following

$$\left\|E_{QM}\right\|^2(m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m}$$

$$\geq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} (-1)$$

$$= (-1) \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} = -1 \tag{68}$$

Clearly, the above inequality can have the lower limit since the following cases are possible:

$$\left\|\{l \mid l \in N \wedge r_l(\sigma_x) = 1\}\right\| = \left\|\{l' \mid l' \in N \wedge r_{l'}(\sigma_x) = -1\}\right\| \tag{69}$$

and

$$\left\|\{l \mid l \in N \wedge r_l(\sigma_x) = -1\}\right\| = \left\|\{l' \mid l' \in N \wedge r_{l'}(\sigma_x) = 1\}\right\| \tag{70}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the realistic theory is true (in a spin-1/2 system), that is

$$-1 \leq \left\|E_{QM}\right\|^2(m) \leq 1 \tag{71}$$

From Strong Law of Large Numbers, we have

$$-1 \leq \left\|E_{QM}\right\|^2 \leq 1 \tag{72}$$

Hence we derive the following proposition concerning the realistic theory

$$\left\| E_{QM} \right\|^2_{\min} = -1 \text{ and} \left\| E_{QM} \right\|^2_{\max} = 1 \tag{73}$$

We cannot accept the two relations (58) (concerning the wave function analysis) and (73) (concerning the realistic theory), simultaneously. Hence we are in the BKS contradiction.

6. Conclusions

In conclusions, recently, a new measurement theorybased on the truth values has been proposed [38]. The resultsof measurements have been either 0 or 1. The measurementtheory has accepted a hidden variables modelfor a single Pauli observable. Therefore we can have introduceda classical probability space for the measurementtheory in this case. On the other hand, we havediscussed the fact that the projective measurement theory(the results of measurements are either +1 or -1)says the Bell, Kochen, and Specker (BKS) paradox forthe single Pauli observable. To justify our assertion, wehave presented the BKS theorem in almost all the twodimensionalstates, by using the projective measurementtheory. As an example, we have presented the BKS theoremin the two-dimensional white noise state, by usingthe projective measurement theory. Our discussion hasprovided new insight to formulate quantum measurementtheory, by using the measurement theory based on thetruth values.

References

[1] J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1955).

[2] J. J. Sakurai, Modern Quantum Mechanics (Addison-Wesley Publishing Company, 1995), Revised ed.

[3] A. Peres, Quantum Theory: Concepts and Methods(Kluwer Academic, Dordrecht, The Netherlands, 1993).

[4] M. Redhead, Incompleteness, Nonlocality, and Realism (Clarendon Press, Oxford, 1989), 2nd ed.

[5] M. A. Nielsen and I. L. Chuang, Quantum Computationand Quantum Information (Cambridge University Press,2000).

[6] R. P. Feynman, R. B. Leighton, and M. Sands, Lectureson Physics, Volume III, Quantum mechanics (Addison-Wesley Publishing Company, 1965).

[7] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47,777 (1935).

[8] J. S. Bell, Physics 1, 195 (1964).

[9] S. Kochen and E. P. Specker, J. Math. Mech. 17, 59(1967).

[10] D. M. Greenberger, M. A. Horne, and A. Zeilinger, inBell's Theorem, Quantum Theory and Conceptions of theUniverse, edited by M. Kafatos (Kluwer Academic, Dordrecht,The Netherlands, 1989), pp. 69-72.

[11] D. M. Greenberger, M. A. Horne, A. Shimony, and A.Zeilinger, Am. J. Phys. 58, 1131 (1990).

[12] C. Pagonis, M. L. G. Redhead, and R. K. Clifton, Phys.Lett. A 155, 441 (1991).

[13] N. D. Mermin, Phys. Today 43(6), 9 (1990).

[14] N. D. Mermin, Am. J. Phys. 58, 731 (1990).

[15] A. Peres, Phys. Lett. A 151, 107 (1990).

[16] N. D. Mermin, Phys. Rev. Lett. 65, 3373 (1990).

[17] A. J. Leggett, Found. Phys. 33, 1469 (2003).

[18] S. Groblacher, T. Paterek, R. Kaltenbaek, C. Brukner,M. Zukowski, M. Aspelmeyer, and A. Zeilinger, Nature(London) 446, 871 (2007).

[19] T. Paterek, A. Fedrizzi, S. Groblacher, T. Jennewein, M.Z_ukowski, M. Aspelmeyer, and A. Zeilinger, Phys. Rev.Lett. 99, 210406 (2007).

[20] C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, and V. Scarani, Phys. Rev. Lett. 99, 210407(2007).

[21] A. Suarez, Found. Phys. 38, 583 (2008).

[22] M. Zukowski, Found. Phys. 38, 1070 (2008).
[23] A. Suarez, Found. Phys. 39, 156 (2009).
[24] D. Deutsch, Proc. Roy. Soc. London Ser. A 400, 97(1985).
[25] D. Deutsch and R. Jozsa, Proc. Roy. Soc. London Ser. A439, 553 (1992).
[26] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, Proc.Roy. Soc. London Ser. A 454, 339 (1998).
[27] J. A. Jones and M. Mosca, J. Chem. Phys. 109, 1648(1998).
[28] S. Gulde, M. Riebe, G. P. T. Lancaster, C. Becher, J.Eschner, H. Haffner, F. Schmidt-Kaler, I. L. Chuang, andR. Blatt, Nature (London) 421, 48 (2003).
[29] A. N. de Oliveira, S. P. Walborn, and C. H. Monken, J.Opt. B: Quantum Semiclass. Opt. 7, 288-292 (2005).
[30] Y.-H. Kim, Phys. Rev. A 67, 040301(R) (2003).
[31] M. Mohseni, J. S. Lundeen, K. J. Resch, and A. M. Steinberg,Phys. Rev. Lett. 91, 187903 (2003).
[32] M. S. Tame, R.Prevedel, M. Paternostro, P. Bohi, M.S. Kim, and A. Zeilinger, Phys. Rev. Lett. 98, 140501(2007).
[33] E. Bernstein and U. Vazirani, Proceedings of theTwenty-Fifth Annual ACM Symposium on Theoryof Computing (STOC '93), pp. 11-20 (1993),doi:10.1145/167088.167097; SIAM J. Comput. 26-5, pp.1411-1473 (1997).
[34] D. R. Simon, Foundations of Computer Science, (1994)Proceedings., 35th Annual Symposium on: 116-123, retrieved2011-06-06.
[35] J. Du, M. Shi, X. Zhou, Y. Fan, B. J. Ye, R. Han, andJ. Wu, Phys. Rev. A 64, 042306 (2001).
[36] E. Brainis, L.-P. Lamoureux, N. J. Cerf, P. Emplit, M.Haelterman, and S. Massar, Phys. Rev. Lett. 90, 157902(2003).
[37] H. Li and L. Yang, Quantum Inf. Process. 14, 1787(2015).
[38] K. Nagata and T. Nakamura, Int. J. Theor. Phys. 55,3616 (2016).
[39] In probability theory, the law of large numbers is a theoremthat describes the result of performing the same experimenta large number of times. According to the law,the average of the results obtained from a large numberof trials should be close to the expected value, andwill tend to become closer as more trials are performed.The strong law of large numbers states that the sampleaverage converges almost surely to the expected value.
[40] K. Nagata and T. Nakamura, Physics Journal, Volume1, Issue 3, 183 (2015).